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COLLEGE ALGEBRA

COLLEGE ALGEBRA

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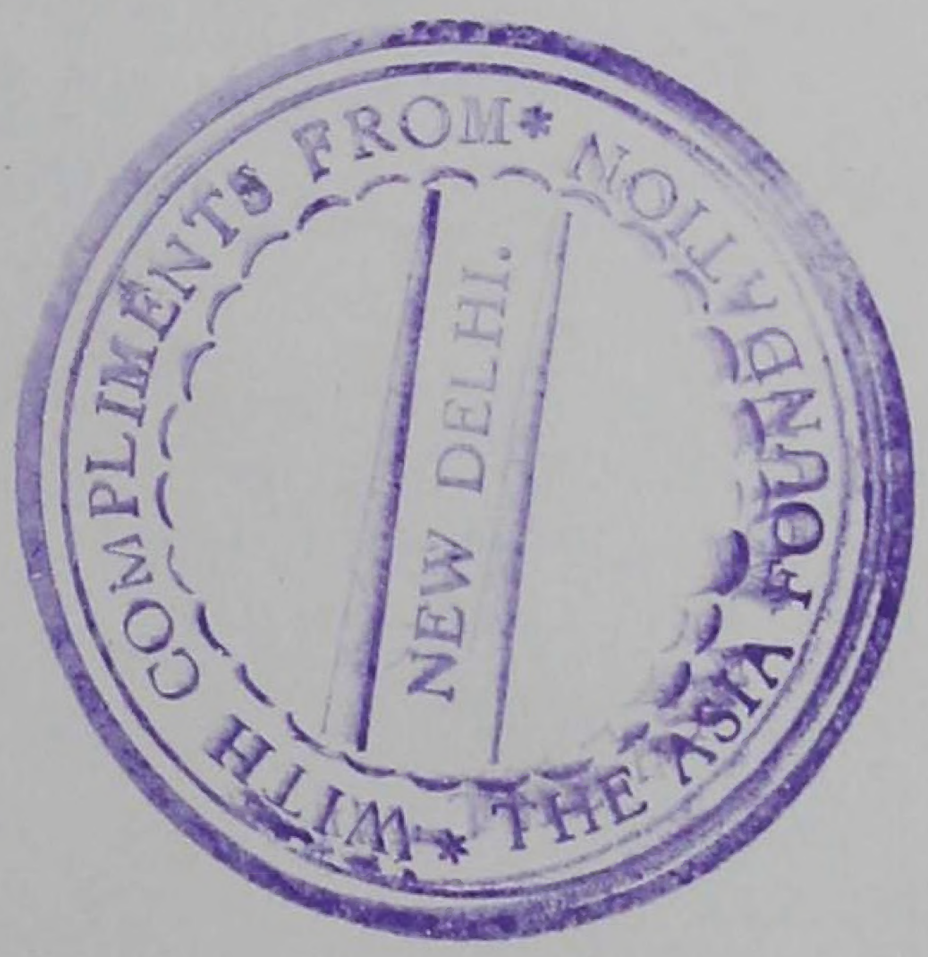

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Preface

SINCE THE GREAT majority of students study algebra primarily to learn its mechanics, this book emphasizes technique in order to meet their needs; but in addition, the book is sufficiently rich in interpretation and general problems to develop the student's powers of analysis. Also, the author has attempted to write an algebra textbook which is sufficiently self-directive so that the student can be independent of the teacher's close supervision. This is accomplished by giving *complete directives* for each algebraic operation, and by giving an abundance of illustrative examples with directives for the complete solution of each.

The twenty-one chapters of this book contain all the conventional material found in most college algebras. The book contains an abundance of material of all kinds with many applications to business and engineering and has been prepared for the first-year students in liberal arts colleges and engineering and technical schools.

The first sixteen chapters contain all the material usually found in any intermediate algebra. In these early chapters the material found in high-school algebra textbooks is presented in a comprehensive manner with much of the related material correlated and presented under a common heading. Even though some of the early chapters may be omitted in a course in college algebra, they nevertheless give the poorer student a means of reviewing his high-school algebra.

At the beginning of most chapters there is a brief outline of the important formulas which are developed in the chapter. At the end of every unit of work, there is a set of questions and a set of exercises. Answers to the odd-numbered exercises are given on page 431. The questions are intended to bring out and stress for the student the important definitions and principles found in the lesson. Through the use of these questions the student can develop self-reliance by checking to see whether he has learned the important facts covered in the lesson.

All technical terms are carefully defined as they are introduced, and stress is laid upon fundamental concepts. These discussions are followed either by illustrations or by examples. The material is presented with logical rigor but no attempt is made to give proofs that are above the level of college freshmen. In such cases a discussion is given but not the proofs.

To add to the cultural maturity of the student, many brief yet complete and accurate historical sketches of elementary mathematics are given. These historical sketches were taken from many reliable sources and they will add interest to the study of algebra and will give the student a better understanding of the development of mathematics.

Whenever possible the material is presented in a sequence that will motivate the student to study the material to follow. For example, in the chapter on Theory of Equations, the theorem on rational roots is given early in the chapter so the student sees that the remaining theorems are given to aid him in solving algebraic equations.

The author wishes to acknowledge his indebtedness and appreciation to the following mathematicians for their many helpful suggestions: to Professor Emeritus E. J. Townsend, formerly Head of the Department of Mathematics at the University of Illinois, and Professor H. R. Brahanna, of the University of Illinois, for reading the manuscript critically; to Professor Emeritus G. A. Miller, of the University of Illinois, for reading the historical notes; to Professor H. P. Rogers, formerly of Kent State University, for using the text in mimeograph form in the classroom; and to his colleagues at Rhode Island State College.

HARRY A. BENDER

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COLLEGE ALGEBRA

CHAPTER I

Our Number System

College algebra consists primarily of the application of the **five fundamental operations** (*addition, subtraction, multiplication, division, and the extraction of roots*) to the numbers in our algebra. These numbers are called **complex numbers.**

The *complex numbers* are divided into *real* and *imaginary numbers.*

The *real numbers* are divided into *rational* and *irrational numbers.*

The *rational numbers* consist of all positive and negative integers, zero, and all positive and negative fractions.

1. POSITIVE INTEGERS

We are all familiar with the process of counting. The numbers so used are called the ***positive integers,*** or ***whole numbers,*** or the ***cardinal numbers.*** Thus 1, 2, 3, . . . , and so on, constitute all the **positive integers** and all the **cardinal numbers.**

The positive integers are used to denote the number of objects or things in a group, but they cannot be used to denote the order in which these objects appear in the group. Hence, if we wish to denote the order in which the objects appear in the group, it is necessary to use a different kind of number, as *first, second, third,* . . . , and so on, which are called the ***ordinal numbers.***

If we apply the **five fundamental operations** (*addition, subtraction, multiplication, division, and the extraction of roots*) to the positive integers, we may obtain numbers that are not positive integers, as -2, $\frac{2}{3}$, $\sqrt{26}$, yet these numbers do have a place in our everyday experiences. Hence, if our algebra is to be complete, it is necessary to include in our number system all numbers that will result when the five fundamental operations are applied, not only to the positive integers, but to any of the resulting numbers.

When **addition** and **multiplication** are applied to the positive integers we do not get any new numbers, for the result is a positive integer.

An integer p is a ***prime*** if p is not 0 or ± 1 and if p is divisible only by ± 1 and $\pm p$.

2. NEGATIVE NUMBERS

Subtracting a larger number from a smaller number leads to a negative number; for example, $5 - 8 = -3$. It is convenient to represent a debit balance by a different kind of number from that used to denote a credit balance. This is done in representing a debit balance by a negative number and a credit balance by a positive number. It is also convenient to represent 20° below zero by the negative number $-20°$ (read $-20°$ above zero, to contrast with $+20°$ above zero.) *A negative number has the opposite meaning to that assigned to a positive number.* That is, if positive numbers are used to represent distances measured to the right, then negative numbers are used to represent distances measured to the left.

3. FRACTIONS

The operation of **division** when applied to the integers may lead to numbers other than the positive and negative integers, as $\frac{3}{4}, \frac{17}{15}, -\frac{4}{11}$, which are called ***fractions.*** *A fraction is a symbol of the form* $\frac{a}{b}$ *which expresses the quotient of two integers, where* a *is called the* ***numerator*** *and* b *the* ***denominator*** *of the fraction.* If a group of objects can be partitioned into smaller groups of equal number, then each smaller group is a fraction of the original group. Fractional results may or may not admit of an interpretation in a particular problem. Thus, 5 boys cannot be separated into two equal groups; however, the equation $2x = 5$ has for its solution $x = \frac{5}{2}$.

4. ZERO

The value of x which satisfies the equation $a + x = a$ is defined to be ***zero,*** thus,

$$a + 0 = a. \tag{1}$$

The number 0 has characteristic properties that no other number has. For example, zero can be added to, or subtracted from, a given number without changing its value, thus,

$$a + 0 = a, \qquad a - 0 = a, \qquad 0 + a = a.$$

But
$$0 - a = -a. \tag{2}$$

If an account of zero dollars is doubled, the account will not be changed; that is, $0 \cdot 2 = 0$. Also,

$$0 \cdot a = 0, \qquad a \cdot 0 = 0. \tag{3}$$

The product of zero and any number is zero.

If zero pennies are divided equally among 20 boys, each boy receives zero pennies; that is, $\frac{0}{20} = 0$. Also,

$$\frac{0}{a} = 0. \qquad (a \neq 0)^* \tag{4}$$

Zero divided by any number different from zero is zero.

However, it is impossible to divide 20 pennies among zero boys. The ratio $\frac{20}{0}$ does not have a numerical value.

It is impossible to divide any number by zero.

By observing the following ratios,

$$\frac{20}{20} = 1, \qquad \frac{20}{10} = 2, \qquad \frac{20}{1} = 20, \qquad \frac{20}{.01} = 2{,}000,$$

we see that the numerical value† of the ratio $\frac{20}{x}$ increases without bound as x is made to approach 0.

It is customary to represent the ratio $\frac{a}{0}$ by the symbol ∞, which is read "infinity"; however, ∞ is not a number. That is, we write

$$\frac{a}{0} = \infty, \tag{5}$$

which means the numerical value of the quotient $\frac{a}{x}$ *is increasing without bound as* x *is made to approach* 0.

The following are a few meaningless operations:

$$\frac{\infty}{\infty}, \quad \frac{0}{0}, \quad 0 \cdot \infty, \quad \text{and} \quad \infty - \infty.$$

* Read "*a is not equal to zero.*"

† By numerical value we mean the value without regard to sign.

However, it is usually possible in each such case to study the limiting value of the expression and thus avoid the meaningless operations. For example, what is the value of $\frac{x^2-4}{x-2}$ when $x = 2$?

Here we have $\frac{4-4}{2-2} = \frac{0}{0}$, which is meaningless. But for all values of x different from 2, $\frac{x^2-4}{x-2} = x + 2$. Hence, we say that the value of $\frac{x^2-4}{x-2}$ approaches 4, the value of $x + 2$, as the value of x approaches 2. For example, when $x = 1.9$, $\frac{x^2-4}{x-2} = \frac{3.61-4}{1.9-2}$ $= 3.9$; when $x = 1.99$, $\frac{x^2-4}{x-2} = 3.99$; and so on. Thus, we write

$$\lim_{x \to 2} \frac{x^2-4}{x-2} = 4, \tag{6}$$

which is read "*the limit of* $\frac{x^2-4}{x-2}$ *as* x *approaches* 2 *is* 4."

5. RATIONAL NUMBERS

A *rational number* is defined to be any number that can be written as the ratio of two integers. Thus, all positive and negative integers, zero, and all positive and negative fractions constitute the system of *rational numbers.*

A rational number can also be defined as any number resulting from applying, a finite number of times, one or more of the following fundamental *rational operations* (*addition, subtraction, multiplication, and division*) to unity (*division by zero excluded*). Furthermore, the four rational operations ($+$, $-$, $\times$, $\div$) when applied to the rational numbers will yield a closed system; that is, the result will be a rational number.*

6. IRRATIONAL NUMBERS

The fifth fundamental operation, the extraction of roots, if applied to the **positive** rational numbers may give rise to real numbers which are different from the rational numbers; for example, $\sqrt{3}$, $\sqrt[3]{5}$, $\sqrt[5]{19}$. Such numbers are called *surds* and are also called *irrational numbers. All the irrational numbers of*

* The proof is omitted in this book.

the form $\sqrt[n]{A}$ *are called surds, where* A *is a* ***positive*** *rational integer not the* n*th power of a rational number* (n *being an integer greater than* 1).

The necessity of including such numbers as $\sqrt{2}$ in our algebra is at once apparent if we wish to find the length of the diagonal of a unit square, that is, a square of sides 1 unit long. We shall show that the number representing the length of this diagonal cannot be represented as the ratio of two integers, and therefore it is not a rational number, but is the irrational number $\sqrt{2}$. To show that $\sqrt{2}$ is not a rational number, we suppose that it is possible to express $\sqrt{2}$ as the ratio of two integers a and b, where the fraction $\frac{a}{b}$ is in its reduced form, and hence a and b do not have a common factor.

Thus, if

$$\sqrt{2} = \frac{a}{b},$$

then after squaring both members of the equation and clearing of fractions we have

$$2b^2 = a^2.$$

This equation cannot be satisfied for a and b integers, because a must be an even integer since $a^2 = 2b^2$, and b must be an odd integer since a and b do not have a common factor. Now a^2 is divisible by 4 and $2b^2$ is only divisible by 2, hence the equation $2b^2 = a^2$ cannot be satisfied for a and b integers. This is contrary to the assumption that a and b were integers.

However, *not all irrational numbers are surds.* For example, the number representing the ratio of the circumference of a circle to its diameter is a real number but it is neither a rational number nor a surd. The irrational number representing this ratio is denoted by the symbol π and it can be shown that this ratio cannot be expressed as the ratio of two integers nor can it be expressed by a finite number of radicals.* Thus, there are two kinds of irrational numbers. All irrational numbers that can be real roots of some algebraic equation (Article 42) with rational coefficients are called ***algebraic irrational numbers*** (this includes all the surds), and the remaining irrational numbers are called ***transcendental numbers.*** The German mathematician G. W. Leibniz (1646–1716), in 1686,

* The proof that π is a transcendental number is beyond the scope of this book.

was the first to define transcendental numbers. The French mathematician J. Liouville (1809–1882), in 1844, was the first to prove the existence of transcendental numbers; that is, to prove the existence of real numbers that cannot be roots of an algebraic equation whose coefficients are rational numbers.

In trigonometry we have other examples of irrational numbers that are transcendental numbers.

Another example of irrational numbers that are transcendental numbers is found in the following illustration. The value of x that satisfies the equation $10^x = 2$ is an irrational number and is called the logarithm of 2 to the base 10. It can be shown, but it is beyond the scope of this book to do so, that the value of x that satisfies this equation cannot be made the root of an algebraic equation with rational coefficients, and hence is neither a rational number nor an algebraic irrational number but is called a transcendental number. The rational number that gives approximately the value of x is 0.301030. This class of irrational numbers will be studied in Chapter V.

7. REAL NUMBERS

A *real number* is defined to be any number that can be written as a decimal, either finite or infinite. Thus, the real numbers consist of all the rational and irrational numbers. *Rational numbers* may be defined as all the real numbers that can be written as a finite decimal and all the real numbers that can be written as a repeating decimal, as 5.34 and 0.242424 · · · . The latter number is the repeating decimal for $\frac{8}{33}$.

An ***irrational number*** may be defined as a real number that cannot be written as a finite decimal or as a repeating decimal. Thus, all the real numbers that are not rational are called irrational numbers.

If on a given straight line we choose a fixed point called the origin, or the zero point, and if we also choose a fixed unit for the measurement of lengths, *then the entire system of real numbers may be represented by the points of the line*, by taking, to correspond to each number, that point whose distance from the origin or zero point is represented by the number. The positive numbers are represented by points to the right of the zero point and the negative numbers are represented by points to the left of the zero point.

It can be shown, but it is beyond the scope of this book to do so, that the distance of any point on the line from the zero point, measured in terms of the fixed unit, is a real number, and that conversely, for every real number, there corresponds a point on the line whose distance from the zero point is represented by this number.

8. IMAGINARY NUMBERS

If we are to admit the five fundamental operations ($+$, $-$, $\times$ $\div$, $\sqrt[n]{\ }$) into our algebra, then it is necessary to include all numbers that may result from these operations when they are applied not only to the positive integers but also to any of the numbers so derived. For example, the square root of a negative number does not appear in our system of rational or irrational numbers.

An even root of a negative number is not a real number but is called an ***imaginary number,*** as $\sqrt{-1}$, $\sqrt[4]{-3}$, $-5\sqrt[6]{-1}$. For brevity, we represent $\sqrt{-1}$ by i, thus, $bi = b\sqrt{-1}$, and we define this to be a ***pure imaginary number*** where b is any real number. If a real number is added to a pure imaginary number, the sum is an imaginary number, as: $-2 + 3\sqrt{-1}$, which can also be written as $-2 + 3i$. Imaginary numbers are not spoken of as being positive numbers or negative numbers, even though the plus and minus signs are used when writing them.

Any number of the form

$$a + bi,$$

where a and b are real numbers, is called a ***complex number.*** All complex numbers are either real numbers or imaginary numbers. For $b = 0$, the complex number $a + bi$ is the real number a; for $b \neq 0$, the complex number $a + bi$ is an imaginary number; and for $a = 0$ and $b \neq 0$, the complex number $a + bi$ is the pure imaginary number bi.

Imaginary numbers are necessary if we are to include numbers in our algebra that will satisfy the equation $x^2 + 1 = 0$. The only values of x that will satisfy this equation are $+i$ and $-i$ ($\pm\sqrt{-1}$).

Our attention should now be called to the fact that, *when the five fundamental operations* ($+$, $-$, $\times$, $\div$, $\sqrt[n]{\ }$) *are applied to the system of complex numbers, the result will be a complex number* (division by zero excepted); that is, under these operations the

system of complex numbers forms a complete or closed number system in our algebra.*

9. GRAPHICAL REPRESENTATION OF COMPLEX NUMBERS

A geometric representation may better serve to picture in our minds the real numbers and the imaginary numbers. The Norwegian surveyor Caspar Wessel (1745–1818), in 1798, was the first to publish a satisfactory graphic representation of the complex numbers. To make such a representation, suppose we divide a plane into four equal parts by two intersecting straight lines $X'X$ and $Y'Y$ perpendicular to each other. See Fig. 1. We may let the points on the horizontal line $X'X$ represent the real numbers, and the points on the vertical line $Y'Y$ represent the pure imaginary numbers; the remaining points in the plane represent the imaginary numbers, if in a given discussion the same unit of length is used to measure all distances. The line $X'X$ is called ***the axis of reals,*** and the line $Y'Y$ is called ***the axis of imaginaries.*** The totality of points in the plane represents all the complex numbers, and the plane is called ***the complex plane.***

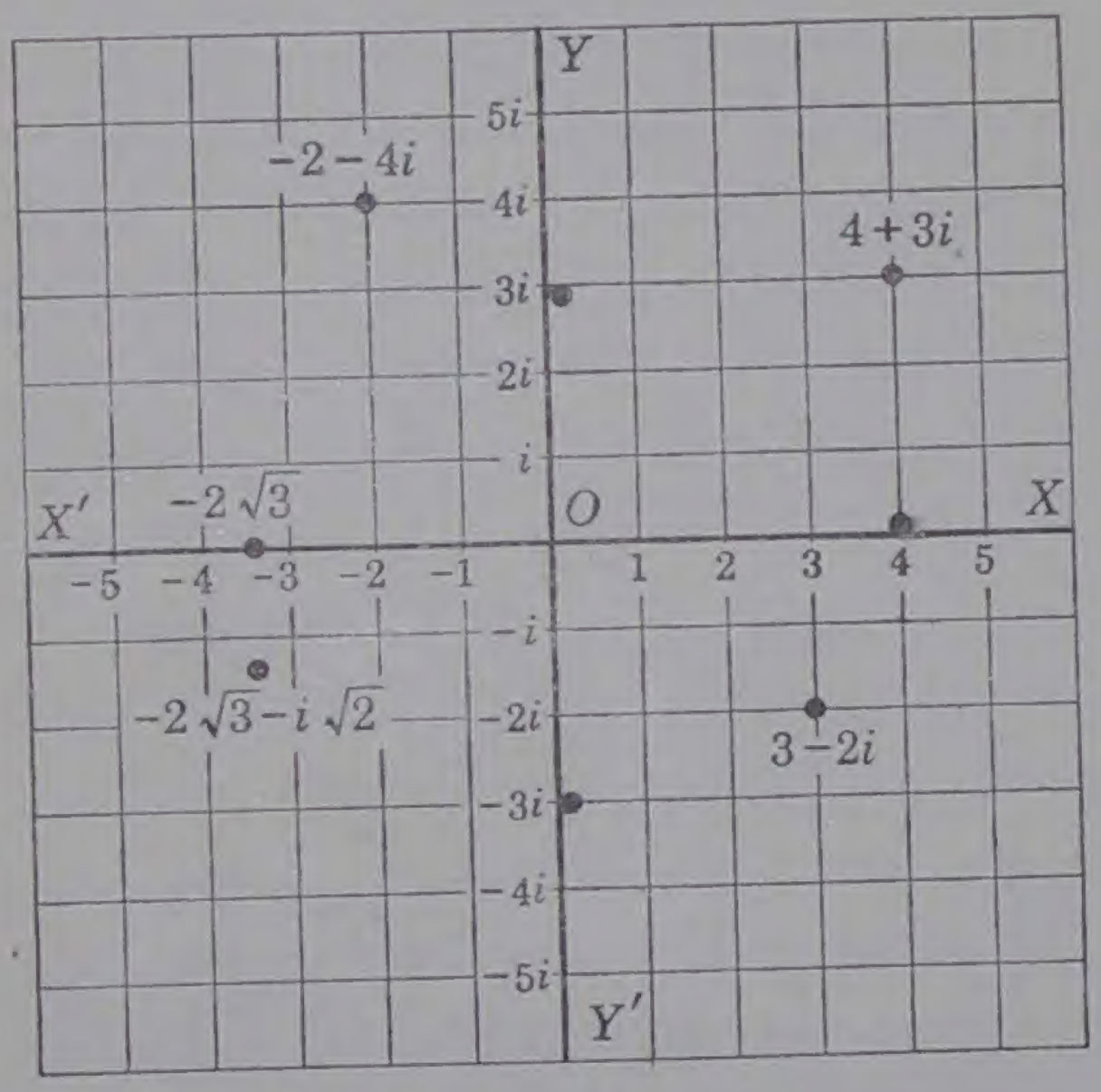


FIG. 1.

The point of intersection of the perpendicular lines represents zero; the points to the right of zero on the horizontal line represent the positive real numbers, and the points to the left of zero on the horizontal line represent the negative real numbers. Thus, the number 4 is represented by a point 4 units to the right measured on the horizontal line, and the number $-2\sqrt{3}$ is represented by a point $2\sqrt{3}$ units to the left measured on the horizontal line.

The pure imaginary number $3i$ is represented by a point 3 units up from 0 on the vertical line, and the pure imaginary number

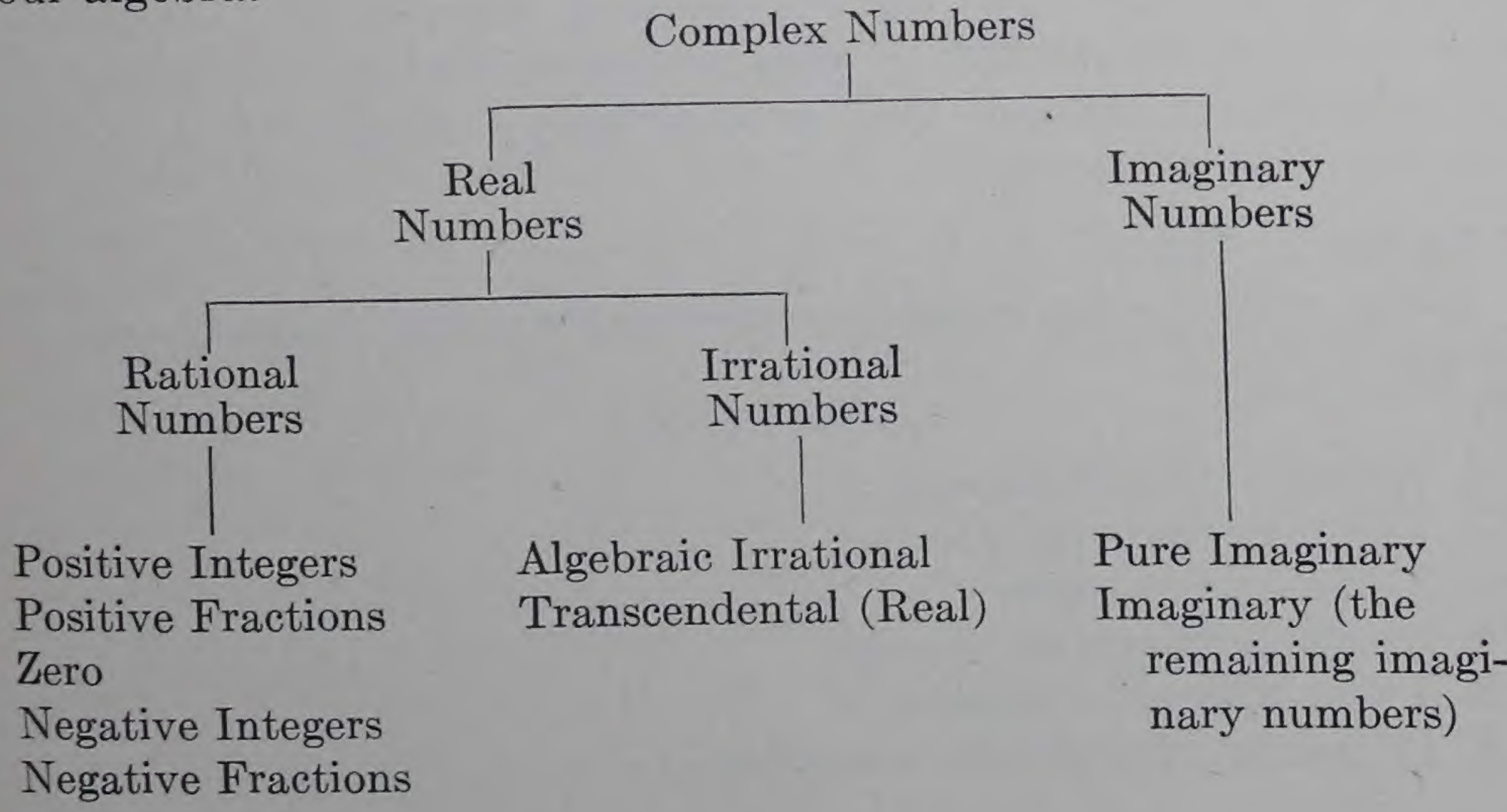
* The proof of this is beyond the scope of this book.

$-3i$ by a point 3 units down from 0 on the vertical line. The imaginary number $4 + 3i$ is represented by a point 4 units to the right of the vertical line and 3 units above the horizontal line. The general complex number $a + bi$ is represented by a point a units from the vertical line, measured to the right if a is positive and to the left if a is negative, and b units from the horizontal line, measured up if b is positive and down if b is negative.

10. SUMMARY

The system of complex numbers is divided into two systems: the system of real numbers and the system of imaginary numbers. These systems are mutually exclusive; thus, if a complex number is not real, it is imaginary, and vice versa. The system of real numbers is divided into two systems: the system of rational numbers and the system of irrational numbers. These systems are also mutually exclusive; thus, all real numbers which are not rational are irrational, and vice versa. The rational numbers are all the positive and negative integers, zero, and all the positive and negative fractions, and the remaining real numbers are called the irrational numbers. The irrational numbers are divided into two classes: algebraic irrational numbers and transcendental numbers. These two classes are also mutually exclusive; thus, all irrational numbers which are not algebraic irrational numbers are transcendental numbers. (Algebraic numbers also include all the rational numbers.)

The following is the classification of the numbers appearing in our algebra.



The concept of positive numbers, including common fractions, and for simple cases the methods of adding, subtracting, multiplying, dividing, and the extracting of roots are very ancient, dating back more than twenty centuries before Christ.

During the development of our number system, many symbols were used from time to time for the ten digits. During certain periods, for centuries at a time, some of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 were used, but would later be replaced by other symbols and then adopted again. It was not until the sixteenth century that all the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 were used for the ten digits, as well as their combinations to represent larger numbers as we use them today.

The signs $+$, $-$, $\times$, $\div$, $\surd$, $=$ were introduced during the fifteenth and sixteenth centuries. However, the sign $\div$ was originally used for subtraction and was used for division for the first time by Rahn, in 1659.

The Greek mathematician Diophantus, who lived during the third century, was the first to mention negative numbers, but he had no conception of the meaning of negative numbers in the abstract, nor did any of the mathematicians for the next thousand years; they all considered negative numbers as absurd numbers and believed it impossible to have numbers less than zero. They knew, however, the laws of operation with negative numbers.

The Italian, Leonardo (Pisano), in 1225, seems to have been the first to give an interpretation of negative results. In a financial problem he interpreted a negative result to mean a loss and not a gain. It was not until the seventeenth century that distances were reckoned as positive or negative, and it really was not until the nineteenth century that a satisfactory theory of negative numbers was set forth.

Example. Classify -9.

The number -9 is a complex number and is a real, rational, negative integer.

QUESTIONS

1. Define a rational number.
2. Define an irrational number.
3. Define a real number.
4. Define an imaginary number.
5. Define a complex number.
6. Define zero and give its properties.

7. Give the classification of complex numbers.
8. How are complex numbers represented graphically?

EXERCISES

Classify each of the following numbers and represent each number graphically.

1. -3.
2. $3 - \sqrt{2}$.
3. $\frac{7}{4}$.
4. $-\sqrt{5}$.
5. 0.
6. $\frac{12}{3}$.
7. $-2\frac{3}{4}$.
8. $\sqrt{25}$.
9. $-2 - 3i$.
10. $\frac{1}{5}\sqrt{5}$.
11. 6.7.
12. $-5 + \sqrt{-2}$.
13. $-4i$.
14. $3.23212621 \cdots$.
15. $2.353535 \cdots$.

CHAPTER II

Fundamental Operations

It is assumed that the methods for carrying out the four rational operations on the positive integers are known. The method for extraction of square roots and cube roots is illustrated by examples in Article 22, Chapter III. It is necessary to define in a consistent manner what is meant by the five fundamental operations when applied to negative numbers, to fractions, and to imaginary numbers. (For imaginary numbers, see Chapter X.)

11. DEFINITION OF ALGEBRA

Our elementary arithmetic consists of the application of the five fundamental operations ($+$, $-$, $\times$, $\div$, $\sqrt[n]{\ }$) to the positive rational numbers only, where the commutative and the associative laws hold for addition, and the commutative, associative, and distributive laws hold for multiplication as explained below.

We may think of college algebra as a gradual generalization of arithmetic governed by the three fundamental laws exhibited in ordinary arithmetic. That is, algebra deals with the general theory of those operations with quantities of which the operations of ordinary arithmetic are a particular case. The quantities consist of numbers and of letters of the alphabet used as general symbols of numbers to which any particular value may be assigned.

The three fundamental laws are the **commutative law,** the **associative law,** and the **distributive law.** The commutative law and the associative law hold for addition and multiplication but do not always hold for subtraction and division.*

* **For subtraction:** (1) The commutative law: $a - b \neq b - a$, given $a \neq b$ (read a not equal to b). (2) The associative law: $a - (b + c) \neq (a - b) + c$, given $c \neq 0$.
For division: (1) The commutative law: $a \div b \neq b \div a$, given $a \neq \pm b$. (2) The associative law: $a \div (bc) \neq (a \div b)c$, given $c \neq \pm 1$. However, (3) the distributive law: $(a + b) \div c = a \div c + b \div c = \frac{a}{c} + \frac{b}{c}$.

For addition:

(1) The commutative law: $a + b = b + a$.

(2) The associative law: $a + (b + c) = (a + b) + c$.

For multiplication:

(1) The commutative law: $a \cdot b = b \cdot a$.

(2) The associative law: $a(bc) = (ab)c$.

Multiplication is distributive with respect to addition; that is,

$(a + b)c = ac + bc$.

12. ADDITION AND SUBTRACTION OF NEGATIVE NUMBERS

If a man has \$5, we say his resources are 5 dollars, and if he is in debt \$5, we say his resources are -5 dollars. Thus, if A's resources are 10 dollars and B's resources are -4 dollars, then their combined resources are 6 dollars. Hence in general symbols we write

$$a + (-b) = a - b \qquad \text{and} \qquad (-b) + a = -b + a, \tag{1}$$

and state the following law or assumption:

Adding the negative number $(-b)$ to the positive number a is equivalent to subtracting the positive number b from a.

Furthermore, A's resources exceed B's resources by 14 dollars, but B's resources exceed A's resources by -14 dollars. Hence in general symbols we write

$$\left.\begin{aligned} a - (-b) &= a + b \\ \text{and} \\ (-b) - a &= -b - a = -(a + b), \end{aligned}\right\} \tag{2}$$

and state the following laws or assumptions:

Subtracting the negative number $(-b)$ from the positive number a is equivalent to adding the positive number b to a.

The result of subtracting the positive number a from the negative number $(-b)$ is the negative number $-(a + b)$.

13. MULTIPLICATION OF NEGATIVE NUMBERS

If A lost 5 dollars and B lost 3 times as much, then B lost 15 dollars; or, if A's resources are -5 dollars and B's resources are 3 times as much, then B's resources are -15 dollars. Hence in general symbols we write

$$(-a)b = -ab \qquad \text{and} \qquad a(-b) = -ab, \tag{3}$$

and state the following law or assumption:

The product of a negative number and a positive number is a negative number whose numerical value is the product of the numerical values of the two numbers. (The numerical value of -15 is 15.)

If A loses 5 dollars a day and now has zero dollars, then 3 days ago he must have had 15 dollars; or we may say if A's resources are changing by -5 dollars per day, then in -3 days his resources will be 15 dollars. Hence, expressed in general symbols we write

$$(-a)(-b) = ab, \tag{4}$$

and state the following law or assumption:

The product of two negative numbers is a positive number whose numerical value is the product of the numerical values of the two numbers.

14. DIVISION OF NEGATIVE NUMBERS

If A's resources are -15 dollars and are divided equally among 3 men, then the resources of each man will be -5 dollars. Hence in general symbols we write

$$\frac{-a}{b} = -\frac{a}{b}, \tag{5}$$

and state the following law or assumption:

The quotient of a negative number and a positive number is a negative number whose numerical value is the quotient of the numerical values of the two numbers.

From the laws of multiplication of negative numbers, and the fact that if $\frac{a}{b} = x$ then $a = bx$, we see that $\frac{15}{5}$, $\frac{-15}{-5}$, $-\frac{-15}{5}$, and $-\frac{15}{-5}$ are each equal to 3, and that $\frac{-15}{5}$, $\frac{15}{-5}$, $-\frac{15}{5}$, and $-\frac{-15}{-5}$ are each equal to -3. Hence the following relations must be true.

$$\frac{a}{b} = \frac{-a}{-b} = -\frac{a}{b} = -\frac{a}{-b}, \tag{6}$$

and

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b} = -\frac{-a}{-b}. \tag{7}$$

Any two of the three parts of a fraction, namely, numerator, denominator, and the fraction, can be multiplied by -1 without changing the value of the fraction. Thus, we can change the sign

of the numerator and denominator, or numerator and fraction, or fraction and denominator, without changing the value of the fraction.

15. ADDITION AND SUBTRACTION OF FRACTIONS

If A has $\frac{1}{2}$ of an apple and B has $\frac{1}{3}$ of an apple, then how much do the two together have? To determine this, let us suppose that an apple has been cut into 6 equal parts and that A has 3 of these parts or $\frac{1}{2}$ of the apple and B has two of these parts or $\frac{1}{3}$ of the apple. Hence A and B together have 5 of these parts, or $\frac{5}{6}$ of an apple. There are two fundamental ideas involved here. The first is that $\frac{1}{2}$ is the same as $\frac{3}{6}$ and $\frac{1}{3}$ is the same as $\frac{2}{6}$. *The numerator and denominator of a given fraction can be multiplied or divided by the same number which is different from zero without changing the value of the fraction.* To illustrate:

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c} = \frac{ac}{bc} \qquad \text{also} \qquad \frac{ac}{bc} = \frac{a}{b}. \tag{8}$$

The second fundamental idea is that $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$. *The algebraic sum of several fractions having the same denominator is a fraction whose numerator is the algebraic sum of the numerators of the given fractions and whose denominator is the denominator that is common to all of the fractions.*

To add or subtract fractions with different denominators, the student will find it advantageous to first express each of the different fractions as equivalent fractions all having the same common denominator. This is accomplished by finding the smallest number or expression, called the **least common denominator,** which contains each of the separate denominators as a factor; then, for each fraction, we multiply the numerator and the denominator by the quotient obtained by dividing the common denominator by the denominator of this fraction. Thus, in $\frac{m}{ac} + \frac{n}{ab}$, abc is the smallest number into which ac and ab can be divided without a remainder; dividing abc by ac the quotient b; multiplying both m and ac by b, we have $\frac{m \cdot b}{ac \cdot b} = \frac{mb}{abc}$. Similarly, $\frac{n}{ab} = \frac{nc}{abc}$. Hence we have

$$\frac{m}{ac} + \frac{n}{ab} = \frac{mb}{abc} + \frac{nc}{abc} = \frac{mb + nc}{abc}. \tag{9}$$

16. MULTIPLICATION OF FRACTIONS

If A has $\frac{1}{2}$ of an apple and B has $\frac{1}{3}$ as much as A, then how much does B have? As before, let us suppose that the apple has been cut into 6 equal parts and that A has 3 of these parts or $\frac{3}{6}$ of an apple; then if B has $\frac{1}{3}$ as much he has 1 part or $\frac{1}{6}$ of an apple. Thus $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$, and for the general case

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}. \tag{10}$$

The product of two or more fractions is a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators of the several fractions.

17. DIVISION OF FRACTIONS

Suppose we wish to separate 6 objects into 3 equal groups. We may divide the 6 objects into 3 equal groups or we may form 3 groups and place in each group $\frac{1}{3}$ of the number of objects. Thus, dividing 6 by 3 is the same as multiplying 6 by $\frac{1}{3}$, or $6 \div 3 = 6 \cdot \frac{1}{3}$. For the general case, we have

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}. \tag{11}$$

To divide one fraction by another fraction, multiply the dividend by the divisor inverted.

18. ADDITION AND SUBTRACTION OF ALGEBRAIC FUNCTIONS

An expression formed from letters and numbers by the aid of the signs of addition, subtraction, multiplication, division, and the extraction of roots, each operation being used a finite number of times, is called an ***algebraic expression,*** *or an* ***algebraic function.***

The addition and subtraction of algebraic functions differ slightly from the addition and subtraction in arithmetic. For example, $12 + 25 = 37$ but the sum $a + b$ can be indicated only as $a + b = a + b$ unless we know more about the numbers a and b. However, $2 \cdot 3 + 5 \cdot 3 = 7 \cdot 3$ and likewise $2a + 5a = 7a$. ($2a$ means $2 \cdot a$.)

COEFFICIENT. *If an algebraic expression is composed of the product of several factors consisting of letters and numbers, then the*

product of some of these factors is the **coefficient** *of the product of the remaining factors.* For example, in $63a^2b(x+y)$, 63 is the coefficient of $a^2b(x+y)$; 9 is the coefficient of $7a^2b(x+y)$; $63a^2b$ is the coefficient of $(x+y)$; $(x+y)$ is the coefficient of $63a^2b$; and so on. *In the simple case of* $x - y$, *the coefficient of* x *is* 1 *and the coefficient of* y *is* -1.

TERM. If an algebraic expression is written as the sum and difference of several expressions, then each of the latter expressions, together with the sign immediately before it, is called a ***term.*** Thus in

$$26 + x - ax^2 + \frac{3ab}{5xy} - \frac{a}{x+y} + \sqrt{x-y} \tag{12}$$

the terms are

$$26, \quad x, \quad -ax^2, \quad \frac{3ab}{5xy}, \quad -\frac{a}{x+y}, \quad \text{and} \quad \sqrt{x-y}. \tag{13}$$

If an algebraic expression is written as

$$3a(x-y) + \frac{a-b}{x+y}, \tag{14}$$

the terms are

$$3a(x-y) \quad \text{and} \quad \frac{a-b}{x+y}, \tag{15}$$

but if this expression is written as

$$3ax - 3ay + \frac{a}{x+y} - \frac{b}{x+y}, \tag{16}$$

the terms are

$$3ax, \quad -3ay, \quad \frac{a}{x+y}, \quad \text{and} \quad -\frac{b}{x+y}. \tag{17}$$

To add or subtract algebraic functions, we add or subtract the ***numerical*** *coefficients of like terms for the new coefficient of the term.* By like terms we mean terms of the form $kx^\alpha y^\beta z^\gamma \cdots$, which are alike with reference to all the letters x, y, z, $\cdots$ and their respective powers. For example,

$$5ab^2 + 6ab - 3ab^2 + 2ab = 2ab^2 + 8ab. \tag{18}$$

The terms $5ab^2$ and $-3ab^2$ are like terms, and $6ab$ and $2ab$ are like terms.

It is more convenient to add several functions if the work is arranged in column form so that like terms are in the same column. Thus, to add $7y^3 - 3xy^2 - 4x^2y + x^3$, $-5x^3 - 11xy^2 - 12xz^2 - y^3$,

$4x^2y - xz^2 - y^2 - 5xy^2$, and $-4xz^2 + y^2 - z^3 + 6xy^2 + 10x^3$, we have, after arranging,

$$\begin{array}{rrrrr} 7y^3 - & 3xy^2 - & 4x^2y + & x^3 & \\ -y^3 - & 11xy^2 & - & 5x^3 - & 12xz^2 \\ - & 5xy^2 + & 4x^2y & - & xz^2 - y^2 \\ & 6xy^2 & + 10x^3 - & 4xz^2 + & y^2 - z^3 \\ \hline 6y^3 - & 13xy^2 & + & 6x^3 - & 17xz^2 - z^3 \end{array} \tag{19}$$

If the coefficients are letters instead of numbers, as $ax + bx$, then this function may be written as $(a + b)x$. This is usually called factoring instead of adding. However, the ordinary addition, as $5ab^2 - 3ab^2 = (5 - 3)ab^2 = 2ab^2$, might be called a special case of factoring.

In case there are no terms alike, as in $(a + b) + (c - d)$, then we can indicate the sum only as $a + b + c - d$.

19. PARENTHESES

Parentheses are used for grouping and hence are used to inclose terms each of which is to be multiplied by the coefficient of the parentheses, as $2(a + b) = 2a + 2b$ and $-(a - b) = (-1)(a) + (-1)(-b) = -a + b$. Thus, the terms within the parentheses are to be regarded as a unit.

Observe that $2(3 \cdot 4)$ equals either $6 \cdot 4$ or $3 \cdot 8$ but not $6 \cdot 8$; that is, $2(3)(4) = (2 \cdot 3)(4)$, or $(3)(2 \cdot 4)$. *To multiply the product of several pairs of parentheses by a given number, multiply each term within but one pair of parentheses by this given number.* To illustrate:

$$3(a + b)(x - y) = (3a + 3b)(x - y)$$

or

$$3(a + b)(x - y) = (a + b)(3x - 3y). \tag{20}$$

In case we wish to reduce the number of pairs of parentheses by one pair, *inclose the product of any two pairs of parentheses by a single pair.* To illustrate:

$$(x + y)(3x - y)(x - 2y) = (3x^2 + 2xy - y^2)(x - 2y)$$

or

$$(x + y)(3x - y)(x - 2y) = (3x - y)(x^2 - xy - 2y^2). \tag{21}$$

And

$$-(x - y)(a^2 + ab - b^2) = (-x + y)(a^2 + ab - b^2)$$

or

$$-(x - y)(a^2 + ab - b^2) = (x - y)(-a^2 - ab + b^2). \tag{22}$$

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Observe that numbers or letters contained between two adjacent signs of operation may be inclosed in parentheses without changing the value of the expression. To illustrate:

$$a \times b - c + d \div e$$

can be written as

$$(a) \times (b) - (c) + (d) \div (e)$$

and thus is

$$(ab) - (c) + (d)\left(\frac{1}{e}\right),$$

or

$$ab - c + \frac{d}{e}.$$

This fact is very important when simplifying problems containing numbers and letters connected by the four signs of operation, $(+, -, \times, \div)$. For we see that *the operations of multiplication and division should be performed before the operations of addition and subtraction.* Observe the errors in the following methods of grouping the foregoing example: $a(b - c) + d \div e$, or $(a \times b - c + d) \div e$. Also, $9 + 5 \times 4 - 16 \div 8 = 9 + 20 - 2 = 27$, and not $\frac{14 \times 4 - 16}{8} = 5$.

From the commutative law it follows that

$$a \div b \times c = a \times c \div b = (a)\left(\frac{1}{b}\right)(c) = \frac{ac}{b}, \qquad (23)$$

and not $\frac{a}{bc}$. However,

$$a \div (b \times c) = \frac{a}{bc}. \qquad (24)$$

Instead of writing one pair of parentheses within another pair, other symbols that have the same meaning as parentheses are used, such as [ ], called *brackets;* { }, called *braces;* and $\overline{\quad}$, called the *bar* or *vinculum.*

When removing parentheses, it is usually desirable: first, to remove the innermost pair, or pairs, and collect like terms; second to remove the next innermost pairs and collect like terms; and so on. Thus,

$$
\begin{aligned}
& 2x - 3[12x - \{x - 4(2x - \overline{3x + 4})\} - 2(5x - 2)] \\
&= 2x - 3[12x - \{x - 4(2x - 3x - 4)\} - 2(5x - 2)] \\
&= 2x - 3[12x - \{x + 4x + 16\} - 2(5x - 2)] \\
&= 2x - 3[12x - 5x - 16 - 10x + 4] \\
&= 2x + 9x + 36 \\
&= 11x + 36. \qquad (25)
\end{aligned}
$$

However, it is not necessary to collect like terms after removing each pair of parentheses; like terms may be collected after all the parentheses have been removed.

QUESTIONS

1. Define algebraic function.
2. Define coefficient.
3. Define term.
4. What is meant by adding or subtracting algebraic expressions?
5. Give the order of performing the operations $+$, $-$, $\times$, $\div$ when they appear in the same expression.

EXERCISES

1. Discuss each of the six following expressions:

(a) $0 \cdot 25$; (b) $-25 \cdot 0$; (c) $0 - 150$;

(d) $\frac{0}{5}$; (e) $\frac{0}{-5}$; (f) $\frac{5}{0}$.

2. Give the terms in each of the following expressions:

(a) $6ab^3x^2 + 5x^3c^2 - 78ab^7 + 5(2x^3a - 7y^2z)$;

(b) $2(x + 3y^2) - \frac{a - b}{a + b} + \sqrt{x^2 - y^2} + \frac{x}{ac - y + z} - \frac{ax - 6by^2 + 7}{5az - 6y^2 + 7}$.

3. Give the value of each of the twelve following expressions:

(a) $-10 - 5$; (b) $-10 - (-5)$; (c) $(-6)(3)(-2)$;

(d) $(-2)(-5)(6)(-3)$; (e) $\frac{(-6)(3)}{(-2)}$; (f) $\frac{(-2)(-6)}{(-5)(-3)}$;

(g) $\frac{5}{4} - \frac{3}{5}$; (h) $\frac{2}{3} - 2$; (i) $\left(\frac{5}{4}\right)\left(-\frac{3}{5}\right)\left(\frac{2}{3}\right)$;

(j) $(-5)\left(\frac{3}{4}\right)$; (k) $\frac{(\frac{3}{5})(-\frac{5}{6})}{(\frac{2}{3})(\frac{5}{9})}$; (l) $\frac{(2)(-3) + (4)(\frac{1}{3})}{\frac{2}{3} - 5}$.

4. For each of these expressions,

$$6ax^2y, \quad \tfrac{3}{4}x^2ay, \quad 4x^2 - 5ay, \quad -3x^2y, \quad ax^2 - 2ya,$$

do the following: (a) Give the coefficient of a. (b) Give the coefficient of x^2. (c) Give the coefficient of y.

5. Find the value of $2 \times 3 - 4 + 6 \div 3$ and of $3 - 2 \times 4 + 10 \div 5 - 3$.

6. Find the value of $12 \div 3 \times 2 - 7 \times 16 \div 2 + 6$ and of $22 - 10 \div 2 + 7 - 36 \div 2 \times 9$.

Collect in parentheses the coefficients of x, y, and z, respectively, in the next two exercises.

7. $3x - 2y + ax - z - 4ay + 3az + 2x$.

8. $-2y - ax + 5z - cy + 2x - 5ay + cz - 3az$.

9. Express each of the following as the product of two pairs of parentheses: $-2(x - y)(a - b)$; $-(x - y)(x + y)(a - b)$; $-(-2)(3)(-4)(5)$.

10. Given the fraction $\dfrac{(x - y)(3x^2 - 2y - 6)}{(x^2 - 5y + 2)(a - b)(c - d)}$. In each of the six following expressions supply the missing part or parts so that the resulting fraction is equivalent to the original fraction.

(a) $-\dfrac{(x - y)(3x^2 - 2y - 6)}{(\quad)(\quad)(\quad)}$;

(b) $-\dfrac{(\quad)(\quad)}{(x^2 - 5y + 2)(a - b)(c - d)}$;

(c) $+\dfrac{(x - y)(6 + 2y - 3x^2)}{(\quad)(\quad)(\quad)}$;

(d) $+\dfrac{(\quad)(\quad)}{(5y - 2 - x^2)(b - a)(d - c)}$;

(e) $-\dfrac{(y - x)(\quad)}{(x^2 - 5y + 2)(b - a)(c - d)}$;

(f) $+\dfrac{(y - x)(6 + 2y - 3x^2)}{(\quad)(b - a)(d - c)}$.

In Exercises 11 through 20, remove parentheses and collect like terms.

11. $a - 3[2 + 3(a - 2b) + 2a] - 3b$.

12. $a - 3[a + 3 - 2(a - \overline{5 + 2a}) + 5\{2 - 3a(2 - \overline{4 - 3})\} + 3]$.

13. $2x - \{3 + 2x - 5[y - 2(x + 4y - 3) + 2x] - 3x - [2 + 5(x - y)] - 5\}$.

14. $2x - 3\{x - 2y + [3x - 7 + 2(x - 3y + \overline{4 - 2x}) - 6x + 2] - 5\} + 6y$.

15. $a - \{2b + 3[c - 2b + 5(a - 3) + 6 - 2(a - b + 3c)]\}$.

16. $3x - 2\{2x - 5 - [x - 4(3x - 2) - 2] - 3x - 2(x - 4) - 5x\} - 3x - 2(x - 3) - 4$.

17. $x - [3x + (2x - 4x)] - 2x - \{3x - [(2x + 3) - 9x]\}$.

18. $2x - 3y + 2[z - 3x + 2 - (3x - 2y + 4z - 5) - y - \overline{4y + 2z - x} + 3 - 2y] + 3z - x - 3(x - 2y + z) - 7.$

19. $3x - 3[2a - (3y - 2) - 2y + 2\{3x - (4a - y - 3x)\}].$

20. $\frac{2}{3} - \frac{3}{4}[\frac{2}{3} - \frac{3}{4}(1\frac{1}{2} - 4\frac{2}{3} \cdot \frac{1}{7}) - 3\frac{2}{3}] + \frac{1}{6}.$

21. Combine like terms in $-5x + 6x^2 - 2x - 3ax + 5x^2$ and in $a + 3ax - 4a - 5a^2x + ax.$

22. Add $3x^2 - x^2y + y^2$, $2xy^2 - 3x^2y^2 - 6y^2$, $5y^2 - 2xy^2 - 7x^2y^2$, $6x^2y^2 - 2xy^2 + 3x^2y$, and $2y^2 - 7x^2y^2 - 2xy^2 + 3x^2y^2 - 5x^2.$

23. Add $3a^2 + 4 - 5b$, $4a + 6b^2 - 3b$, $5a + 2b^2 - 9$, $2a^2 - 2a - 7$, and $2b - 3b^2 + a^2.$

24. Add $3ax - a^2x + 2x^3a$, $-5ax^2 + 2x^3a - 3xa^2 + 2ax$, $3a^2x - 2xa^3 - 4ax^3 + 5x^2a$, and $3a^3x - 4x^3a - 2ax^2 + xa - 5a^2x.$

25. Add $x^2 - ax + 3x - 5y$, $x - y$, $7y + 2x^2 + 3ax$, and $-3x - 3x^2 - 2ax.$

26. Add $6a^2 - 2x^2 + 3y^2$, $3ab + 2xy - 3ay$, $4a^2 + yx - 2ya$, $3x^2 - 2xy - 7y^2$, $6ab - 7ax - 2y^2$, and $2xy + 3ay - 4ax.$

27. From $5a - 6ab - 7a^2 - 2b$, subtract $-2a + 3b - 6a^2 + 7ab.$

28. From $3ax - 2x^2a + 5ax^3 - 7a^2x$, subtract $5xa + 6a^2x - 3xa^3 + 2x^2a.$

29. Subtract $3xy - 6x + 2y - 5$ from $9x - 4y + xy - 2.$

30. From $a^4 - 3a^3b + 2a^2b^2 - 4ab^3 + 4b^4$, subtract $3b^4 + 2ab^3 - 7a^2b^2 + 2a^3b + 7a^4.$

31. From $3x^2y + 2x^3y^2 - 5xy^2$, subtract $2x^2y - 3x^3y^2 - 4xy^2.$

32. Subtract $6a^2 - 2xy + 3x^2y - 4y^2 + 7ay - 2yb + 7xy^2$ from $3x^2 - 4ay + 2y^2x - 3xy - 7a^2 + 2by - 6yx^2.$

33. Subtract $ax^2 - a^2x - ax + a^2x^2$ from $ax - a^2x^2 - a^2x + ax^2.$

34. Subtract $2x^2 - 3xy + 4y^2 - 6y + 7$ from the sum of $2y^2 - x^2 + 3x - 4y - 10$ and $2x^2 - xy - 4y^2 + 2y - 3x + 7.$

35. From the sum of $6a - 5b + 4c - 2ab + 3ac - 5bc$ and $2c - 3b + 4bc - 6a - 3ac + 3ab - 7$, subtract $2a - 4b + 6ab - 7c + 10.$

36. Subtract $\frac{7}{4}x^2 - \frac{2}{3}xy + \frac{9}{2}y^2 - \frac{5}{3}x - \frac{3}{4}y + \frac{11}{3}$ from the sum of $\frac{2}{3}x^2 - \frac{3}{4}y^2 + 7x - \frac{2}{5}y + \frac{7}{2}$ and $\frac{5}{2}x^2 - \frac{3}{4}xy + \frac{7}{3}y^2.$

CHAPTER III

Exponents and Radicals

LAWS OF OPERATION

I. $a^m \cdot a^n = a^{m+n}$.

II. $\dfrac{a^m}{a^n} = a^{m-n}$.

III. $(a^m)^n = a^{mn}$.

IV. $(ab)^n = a^n b^n$.

V. $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$.

VI. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

VII. $\sqrt[n]{a^m} = \sqrt[nk]{a^{mk}}$.

DEFINITIONS

1. $a^{1/n} = \sqrt[n]{a}$.

2. $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$.

3. $a^{-n} = \dfrac{1}{a^n}$.

4. $a^0 = 1$.

20. EXPONENTS AND POWERS

It is convenient to represent the product of several numbers that are the same by using that number once with the appropriate exponent. Thus, $5 \cdot 5 \cdot 5 \cdot 5 = 5^4$. *We shall define* a^m when m is *a positive integer greater than* 1, *to be the product of* m a*'s* (*for* $m = 1$, $a^1 = a$). Thus,

$$a^m = a \cdot a \cdot a \cdot a \cdot a \cdot \cdots a \qquad (m \text{ factors in all}). \qquad (1)$$

The symbol a^m is read "*a to the mth power*" or "*a exponent m*" and is called "*the mth power of a.*" The integer m is called the ***exponent*** of the ***power*** a^m. In 5^4, the exponent of 5 is 4.

21. LAWS OF EXPONENTS

Multiplication and division of algebraic expressions differ slightly from multiplication and division in arithmetic. For example, $5 \cdot 3 = 15$, but in algebra sometimes we can only indicate the operation and at other times we can carry it out. In case all the letters are different, we can only indicate the multiplication

and division, as $(a \cdot b \cdot c) \div (d \cdot e \cdot f) = \dfrac{abc}{def}$. In case some of the letters are repeated, we have the following laws of combination for multiplication, division, and powers.

Laws of Operation

The product of $a^3 \cdot a^4 = (a \cdot a \cdot a)(a \cdot a \cdot a \cdot a) = a^{3+4} = a^7$. For positive integral values of m and n, $\boldsymbol{a^m \cdot a^n}$ *is the product of* $\boldsymbol{m}$ $\boldsymbol{a}$*'s by* $\boldsymbol{n}$ $\boldsymbol{a}$*'s, or the product of* $(\boldsymbol{m+n})$ $\boldsymbol{a}$*'s.* Thus, we can write Law I:

(I) $$\boldsymbol{a^m \cdot a^n = a^{m+n}.}$$

The product of the **m***th power of a number by the* **n***th power of the* **same** *number is equal to the* $(\boldsymbol{m+n})$*th power of this number.*

If we divide a^5 by a^3 we have $\dfrac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a} = a \cdot a \cdot 1 \cdot 1 \cdot 1 = a^2$; or that is, the quotient is the product of as many a's as the number of a's in the numerator exceeds the number of a's in the denominator. For positive integral values of m and n, $\boldsymbol{a^m \div a^n = \dfrac{a^m}{a^n}}$ *is the product of as many* $\boldsymbol{a}$*'s as* $\boldsymbol{m}$ *exceeds* $\boldsymbol{n}$**.** (It is necessary to assume $m > n$ when deriving this law; however, together with the definitions to be given later, this law is valid for any rational values of m and n.) Thus we can write Law II:

(II) $$\boldsymbol{a^m \div a^n = \frac{a^m}{a^n} = a^{m-n}.}$$

The quotient of the **m***th power of a number divided by the* **n***th power of the* **same** *number is equal to the* $(\boldsymbol{m-n})$*th power of this number.*

The value of $(a^3)^4$ according to definition is $a^3 \cdot a^3 \cdot a^3 \cdot a^3$ and by Law I equals $a^{3+3+3+3} = a^{12}$. For positive integral values of m and n, $(a^m)^n = a^m \cdot a^m \cdot a^m \cdot \cdots a^m$ (n factors), which is the product of as many a's as the sum of n m's. Thus, we can write Law III:

(III) $$\boldsymbol{(a^m)^n = a^{mn}.}$$

The **n***th power of the* **m***th power of a number is the* **mn***th* (**m** times **n***th*) *power of this number.*

Since $(ab)^n = ab \cdot ab \cdot ab \cdot \cdots ab$ (n factors) is the product of n a's and n b's, we can write Law IV:

(IV) $$\boldsymbol{(ab)^n = a^n b^n.}$$

*The **n**th power of a product of two or more factors is equal to the product of the **n**th power of each factor.*

Observe that $(a \pm b)^n \neq a^n \pm b^n$; that is, the nth power of a sum or difference does not equal the sum or difference of the nth power of each number. To illustrate, $(3 + 4)^2 = 7^2 = 49$ and not $3^2 + 4^2 = 9 + 16 = 25$.

Definitions

The definition for a^m given in Article 20 becomes meaningless when m is negative or is a fraction. Thus, it is necessary to define a^m differently for negative or fractional values of m. Since the foregoing ***laws of operation*** were developed for positive integral exponents only, it would be convenient if a meaning could be assigned to a^m for negative and fractional values of m so that the same laws of operation would hold for negative and fractional exponents as hold for positive integral exponents.

If we wish Law III to hold when $m = \frac{1}{n}$, then it is necessary that $(a^{1/n})^n = a^{n/n} = a$. Hence, $a^{1/n}$ must be a number that when raised to the nth power gives a. Therefore, we define $a^{1/n}$ to be an nth root of a so that the foregoing laws of operation hold for fractional exponents, and by definition we write

$$a^{1/n} = \sqrt[n]{a}. \tag{1}$$

*Thus, $a^{1/n}$ is defined to be the **n**th root of **a*** (n being a positive integer).

In Article 68, Chapter XII, we shall see that the equation $x^n - a = 0$ has n roots, real or imaginary, and hence, if a is different from zero, there are n distinct nth roots of a. However, *we shall define $\mathbf{a^{1/n}}$ to be single valued and the **principal nth root of a** if **a** is positive, or if **a** is negative and **n** is odd.*

*The principal **n**th root of **a**, when **a** is positive and **n** is any positive integer, is defined to be the positive **n**th root of **a**.*

*The principal **n**th root of $-\mathbf{a}$, when **a** is positive and **n** any positive odd integer, is defined to be the negative **n**th root of **a**.*

No attempt is made to define any of the nth roots of a negative number to be a principal nth root when n is even, nor do we attempt to define any of the nth roots of an imaginary number to be a principal nth root.

For negative numbers we thus have

$$(1') \qquad (-a)^{1/n} = -(a)^{1/n} = \sqrt[n]{-a} = -\sqrt[n]{a}$$

(for n odd and $a > 0$). If n is even and a positive, then $(-a)^{1/n}$ is imaginary and is not defined to be single valued but will be studied in Article 54, Chapter X.

From Law III,

$$a^{m/n} = (a^m)^{1/n} = (a^{1/n})^m.$$

Then by Definition 1 we have

$$(2) \qquad a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

For positive values of a, $a^{m/n}$ *is defined to be the* n*th root of the* m*th power of* a, *or the* m*th power of the* n*th root of* a. The fractional exponent $\frac{m}{n}$ should be reduced to its simplest form; that is, m and n should not have a common factor other than 1.

For fractional powers of negative numbers, we have

$$(2') \qquad (-a)^{m/n} = \sqrt[n]{(-a)^m} = (\sqrt[n]{-a})^m$$

for n odd, but if n is even and a is positive, $(-a)^{m/n}$ is imaginary and is not defined to be single valued.

Next consider what meaning must be assigned to a^m for m a negative number, if the four laws of operation are to hold for negative exponents. If Law I is to hold for negative exponents, we have $a^m \cdot a^{-n} = a^{m-n}$, and from Law II, we also have $a^m \cdot \frac{1}{a^n} = a^{m-n}$. Hence it is necessary to define a^{-n} to be equal to $\frac{1}{a^n}$ since their products by a^m are equal. Therefore,

$$(3) \qquad a^{-n} = \frac{1}{a^n}; \quad \frac{1}{a^{-n}} = a^n, \quad \text{and} \quad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n. \quad \begin{matrix}(a \neq 0)\\(b \neq 0)\end{matrix}$$

Thus, a^{-n} *is defined to be the reciprocal of* a^n.

The symbol a^0 still remains to be defined. Since $1 = \frac{a^n}{a^n} = a^{n-n} = a^0$, we define it as

$$(4) \qquad a^0 = 1. \qquad (a \neq 0)$$

The zeroth power of any number different from zero is defined to be 1, *as:*

$$5^0 = 1, \qquad (2)^0 = 1, \qquad \text{or} \qquad (\sqrt[3]{x} + 1)^0 = 1. \tag{2}$$

With the four definitions (1), (2), (3), and (4) established for the meaning of a^m for m a fraction, or a negative number, or zero, the four laws of operation (I), (II), (III), and (IV) hold for any rational values of m and n. Thus, with these four definitions we can use the four laws of operation whether the exponents are integers or fractions and whether they are positive or negative.

Some of the laws of operating with exponents were known to the Greek mathematician Archimedes (287–212 B.C.), although at that time exponents were not used; other representations were used to express the exponent idea. Chuquet, a Frenchman, in 1484, used what is equivalent to our negative exponents as well as the zero exponent, but without having a clear conception of their meaning. In 1637 the French mathematician René Descartes (1596–1650) was the first to use exponents. The English mathematician John Wallis (1616–1703), in 1655, was the first writer to set forth with any completeness the meaning of negative and fractional exponents but did not actually write them. It was the English mathematician Sir Isaac Newton (1642–1727) who, in 1666, introduced the general literal exponent notation.

22. RADICALS AND SURDS

In the expression $b\sqrt[n]{a}$, the symbol $\sqrt[n]{a}$ is called the **radical;** the sign $\sqrt{\ }$ is called the **radical sign;** the integer n is called the **index** of the radical; the number a which is under the radical sign is called the **radicand;** and b is the **coefficient** of the radical.

If $x^n = a$, *then* x *is called* ***an nth root of a,*** n *being any positive integer.*

Since

$$(2)^2 = 4 \qquad \text{and} \qquad (-2)^2 = 4, \tag{3}$$

we see that there are two square roots of 4, either $+2$ or -2. However, *the radical sign* $\sqrt{\ }$ *does not carry with it both the plus and minus sign;* that is,

$$\sqrt{4} = 2 \qquad \text{and} \qquad -\sqrt{4} = -2. \tag{4}$$

To illustrate, if

$$x^2 = 4,$$

then we write

$$x = \pm \sqrt{4},$$

or

$$x = + \sqrt{4} \quad \text{or} \quad x = - \sqrt{4}.$$

Then

$$x = +2 \quad \text{or} \quad x = -2. \tag{5}$$

Addition of Radicals

First simplify each radical, then find the algebraic sum of the coefficients of like radicals for the new coefficient of this radical. By like radicals we mean radicals with the same index and the same radicand. In case some of the radicals are unlike, we merely indicate the sum.

Example 1. Add $2\sqrt{6} + 3\sqrt{2} - 3\sqrt{24} + \sqrt{8}$.

Solution.

$$\begin{aligned} 2\sqrt{6} + 3\sqrt{2} - 3\sqrt{24} + \sqrt{8} &= 2\sqrt{6} + 3\sqrt{2} - 3\sqrt{4 \cdot 6} + \sqrt{4 \cdot 2} \\ &= 2\sqrt{6} + 3\sqrt{2} - 6\sqrt{6} + 2\sqrt{2} \\ &= -4\sqrt{6} + 5\sqrt{2}. \end{aligned}$$

This example may be simplified by replacing each term by a rational number which approximates this term, then finding the sum of these rational numbers, as:

$$\begin{aligned} &2\sqrt{6} + 3\sqrt{2} - 3\sqrt{24} + \sqrt{8} \\ &\quad = 2(2.449) + 3(1.414) - 3(4.899) + (2.828) \\ &\quad = 4.898 + 4.242 - 14.697 + 2.828 \\ &\quad = -2.729. \end{aligned}$$

An approximation to each of these radicals can be found in Table I, page 407.

Multiplication of Radicals

From the laws of exponents, $\sqrt[n]{a} \cdot \sqrt[n]{b} = a^{1/n}b^{1/n} = (ab)^{1/n} = \sqrt[n]{ab}$. Hence

(V) $$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab};$$

except when both a and b are negative and n is even.

The product of two or more radicals with the ***same index n*** *and with* ***positive*** *radicands is equal to the* ***n****th root of the product of the radicands.*

Law V *holds for both positive and negative radicands if* **n** *is odd;* however, Law V *is not true for negative radicands when* **n** *is even.*

To illustrate:

$$\sqrt{-a}\cdot\sqrt{-a} = (\sqrt{-a})^2 = [(-a)^{1/2}]^2 = -a \qquad (a > 0) \qquad (6)$$

and does not equal

$$\sqrt{(-a)(-a)} = \sqrt{a^2} = a. \qquad (7)$$

Since $\sqrt{-a} = i\sqrt{a}$, we have

$$(\sqrt{-a})^2 = (i\sqrt{a})^2 = i^2a = -a. \qquad (8)$$

Likewise,

$$\begin{aligned}\sqrt{-a}\cdot\sqrt{-b} &= i\sqrt{a}\cdot i\sqrt{b}\\ &= i^2\sqrt{ab}\\ &= -\sqrt{ab} \qquad (a > 0, b > 0) \qquad (9)\end{aligned}$$

but

$$\sqrt[3]{-a}\cdot\sqrt[3]{-b} = (-\sqrt[3]{a})(-\sqrt[3]{b}) \quad \text{or} \quad \sqrt[3]{(-a)(-b)} = \sqrt[3]{ab}. \qquad (10)$$

Division of Radicals

From the laws of exponents, we have

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{a^{1/n}}{b^{1/n}} = \left(\frac{a}{b}\right)^{1/n} = \sqrt[n]{\frac{a}{b}}.$$

Hence,

(VI)
$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

The quotient of two radicals with the **same index n** *and with positive or negative radicands is equal to the* **n***th root of the quotient of the two radicands.*

Radicals cannot be multiplied (or divided) by multiplying (or dividing) the radicands when the indices are different.

Laws V and VI are also true when read from right to left; that is:

The **n***th root of a product of several factors is the product of the* **n***th roots of these factors, and*

The **n***th root of a quotient of two numbers is the quotient of the* **n***th roots of these numbers.*

However, *the nth root of a sum or a difference does not equal the sum or difference of the nth roots;* that is,

$$\sqrt[n]{a \pm b} \neq \sqrt[n]{a} \pm \sqrt[n]{b}. \tag{11}$$

To multiply or divide radicals with different indices, it is necessary to express all radicals with the same index. This can be done by observing that $\sqrt[n]{a^m} = a^{m/n} = a^{mk/nk} = \sqrt[nk]{a^{mk}}$, and hence,

(VII) $$\sqrt[n]{a^m} = \sqrt[nk]{a^{mk}}.$$

The exponents of the factors in the radicand can be either multiplied (or divided) by the same constant without changing the value of the radical provided the index is multiplied (or divided) by this constant.

Example 2. Find the product $\sqrt{3} \cdot \sqrt[3]{2}$.

Solution. Since

$$\sqrt{3} = \sqrt[6]{3^3} \quad \text{and} \quad \sqrt[3]{2} = \sqrt[6]{2^2},$$

then,

$$\sqrt{3} \cdot \sqrt[3]{2} = \sqrt[6]{3^3} \cdot \sqrt[6]{2^2} = \sqrt[6]{3^3 \cdot 2^2} = \sqrt[6]{27 \cdot 4} = \sqrt[6]{108}.$$

Example 3. Express $3\sqrt[3]{5}$ under a common radical.

Solution.

$$3\sqrt[3]{5} = \sqrt[3]{3^3} \cdot \sqrt[3]{5} = \sqrt[3]{27 \cdot 5} = \sqrt[3]{135}.$$

To solve equations of the form

$$x^{m/n} = a \tag{12}$$

for x, observe that the $\frac{n}{m}$th power of $x^{m/n}$ is x. Thus, taking the $\frac{n}{m}$th power of both members of Equation (12), we have

$$(x^{m/n})^{n/m} = a^{n/m}; \tag{13}$$

therefore,

$$x = a^{n/m}. \tag{14}$$

Example 4. Solve $(x)^{-4/3} = 81$ for x.

Solution.

$$(x^{-4/3})^{-3/4} = (81)^{-3/4},$$

hence,

$$x = (3^4)^{-3/4} = 3^{-3} = \frac{1}{27}.$$

LAWS OF OPERATION

I. $a^m a^n = a^{m+n}$; example, $x^4x^{-6} = x^{-2}$.

II. $a^m \div a^n = \dfrac{a^m}{a^n} = a^{m-n}$; example, $\dfrac{x^{-3}}{x^{-7}} = x^4$.

III. $(a^m)^n = a^{mn}$; example $(x^{-\frac{3}{5}})^6 = x^{-4}$.

IV. $(ab)^n = a^n b^n$; example, $(8x^{\frac{3}{5}}y^{-4})^{\frac{1}{3}} = 8^{\frac{1}{3}}x^{\frac{3}{5}}y^{-\frac{4}{3}} = 2x^{\frac{3}{5}}y^{-\frac{4}{3}}$.

V. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$ (given $a \geqq 0$); example, $\sqrt[5]{4} \cdot \sqrt[5]{6} = \sqrt[5]{24}$.

VI. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$; example, $\dfrac{\sqrt[4]{33}}{\sqrt[4]{3}} = \sqrt[4]{11}$.

VII. $\sqrt[n]{a^m} = \sqrt[nk]{a^{mk}}$; example, $\sqrt[3]{x^2} = \sqrt[9]{x^6}$.

DEFINITIONS

1. $a^{1/n} = \sqrt[n]{a}$; example, $16^{\frac{1}{4}} = \sqrt[4]{16} = 2$.

1′. $(-a)^{1/n} = -\sqrt[n]{a}$ (given n odd, $a \geqq 0$);
example, $(-8)^{\frac{1}{3}} = -2$, but $(-16)^{\frac{1}{4}}$ is imaginary.

2. $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$; example, $64^{\frac{5}{6}} = (\sqrt[6]{64})^5 = 2^5 = 32$.

2′. $(-a)^{m/n} = \sqrt[n]{(-a)^m} = (\sqrt[n]{-a})^m$ (given n odd);
example, $(-8)^{\frac{2}{3}} = (\sqrt[3]{-8})^2 = (-2)^2 = 4$.

3. $a^{-n} = \dfrac{1}{a^n}$; $\dfrac{1}{a^{-n}} = a^n$; $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$;
example, $3x^{-5} = \dfrac{3}{x^5}$; and $\left(\dfrac{a^2}{b^3}\right)^{-3} = \left(\dfrac{b^3}{a^2}\right)^3 = \dfrac{b^9}{a^6}$.

4. $a^0 = 1$; example, $5x^0 = 5$.

Finding the Square Root

The method for finding the square root of a given number is illustrated by the following example.

Example 5. Approximate the (principal) square root of 698.4 to the nearest hundredth.

Solution.

$$\begin{array}{lll}
 & 2\ \ 6.\ 4\ \ 3 & \\
2\overline{)6\ 98.40\ 00} & & \\
\quad 4 & & (2 \cdot 2 = 4) \\
46\overline{)2\ 98} & 2 \cdot 2 = 4 & \\
\quad 2\ 76 & & (46 \cdot 6 = 276) \\
524\overline{)\ 22\ 40} & 26 \cdot 2 = 52 & \\
\quad 20\ 96 & & (524 \cdot 4 = 2096) \\
5283\overline{)\ 1\ 44\ 00} & 264 \cdot 2 = 528 & \\
\quad 1\ 58\ 49 & & (5283 \cdot 3 = 15849)
\end{array}$$

Therefore $\sqrt{698.4} = 26.43$ (correct to the nearest hundredth).

Finding the Cube Root

The method for finding the cube root of a given number is illustrated by the following example.

Example 6. Find the principal cube root of 42,144,192.

Solution.

$$
\begin{array}{rrl}
 & & \ 3\quad 4\quad 8 \\
3^2 = 9 & & 9\overline{)\,42,144,192} \\
 & 27 & (9 \cdot 3 = 27) \\
300 \cdot 3^2 = 2700 & 15\ 144 & \\
30 \cdot 3 \cdot 4 = 360 & & \\
4^2 = 16 & & \\
\overline{3076} & 12\ 304 & (3{,}076 \cdot 4 = 12{,}304) \\
 & 2\ 840\ 192 & \\
300 \cdot 34^2 = 346800 & & \\
30 \cdot 34 \cdot 8 = 8160 & & \\
8^2 = 64 & & \\
\overline{355024} & 2\ 840\ 192 & (355{,}024 \cdot 8 = 2{,}840{,}192)
\end{array}
$$

Therefore $\sqrt[3]{42,144,192} = 348$.

This is apparent from the fact that if the given number is $(10a + b)^3$, then

$$
\begin{aligned}
(10a + b)^3 &= 1{,}000a^3 + 300a^2b + 30ab^2 + b^3 \\
&= 1{,}000a^3 + (300a^2 + 30ab + b^2)b.
\end{aligned}
$$

QUESTIONS

1. Define exponent.
2. Give the seven laws of exponents.
3. Give the four definitions pertaining to exponents.
4. When can radicands be multiplied?
5. When can radicands be divided?
6. When can radicands be added?
7. How may the equation $x^{m/n} = a$ be solved for x?

EXERCISES

1. Perform the indicated operations in the following twenty-five expressions:

(a) $a^7 \cdot a^3$; (b) $\dfrac{a^7}{a^3}$; (c) $\dfrac{a^3}{a^7}$; (d) $(a^7)^3$; (e) $(a^3)^7$;

(f) $x^{3a} \cdot x^3$; (g) $x^{3a} \cdot x^a$; (h) $\dfrac{x^{3a}}{x^3}$; (i) $\dfrac{x^{3a}}{x^a}$; (j) $(x^{3a})^3$;

(k) $64^{1/6}$; (l) $64^{1/3}$; (m) $64^{1/2}$; (n) $64^{2/3}$; (o) $\left(\frac{1}{64}\right)^{5/6}$;

(p) $64^{-1/6}$; (q) $25^{-3/2}$; (r) $81^{3/4}$; (s) $27^{-4/6}$; (t) $27^{-5/3}$;

(u) $\sqrt{1}$; (v) $\sqrt[3]{1}$; (w) $1^{-1/4}$; (x) $\frac{1}{\sqrt{1}}$; (y) $\frac{1}{1^0}$.

2. Perform the indicated operations in the following twenty-five expressions:

(a) $x^{3/4} \cdot x^{-4/3}$; (b) $x^{a+b} \cdot x^{a-b}$; (c) $x^{5/6} \div \sqrt[3]{x^2}$; (d) $\sqrt[3]{\sqrt{x}}$;

(e) $\frac{a^{1/2}}{a^{1/3}}$; (f) $\frac{x^2}{x^{-3}}$; (g) $\sqrt[3]{x^{3/4}}$; (h) $\sqrt[3]{x^{-4/5}}$;

(i) $x^{-1/2} \cdot x^{1/2}$; (j) $(a^6)^{1/2} \div (a^3)^{2/3}$; (k) $\frac{4x^{-2/3}y^{-3/4}}{5x^{1/6}y^{-7/8}}$; (l) $\frac{2^{-1}a^{-2}b^{-3}}{3a^3b^2}$;

(m) $2^3 \cdot 3^3$; (n) $2^3 \cdot 3^2$; (o) $2^a \cdot 3^a$; (p) $\left(\frac{1}{25}\right)^{3/2}$;

(q) $\left(\frac{4}{9}\right)^{3/2}$; (r) $\left(\frac{1}{8}\right)^{-1/3}$; (s) $32^{-3/5}$; (t) $\sqrt[3]{-64}$;

(u) $(0.01)^3$; (v) $(0.01)^{3/2}$; (w) $(0.001)^{2/3}$; (x) $(10{,}000)^{-3/4}$;

(y) $(0.0001)^{3/8}$.

3. Perform the indicated operations in the following six expressions:

(a) $(-64)^{1/3}$; (b) $(1{,}728)^{3/6}$; (c) $(0.0081)^{-3/4}$;

(d) $(-64)^{2/3}$; (e) $(729)^{1/6}$; (f) $(0.000064)^{5/6}$.

4. Perform the indicated operations in the following six expressions:

(a) $(-64)^{1/2}$; (b) $(64)^0$; (c) $(x^2y^{-3})^0$;

(d) $(225)^{3/6}$; (e) $\left(\frac{16}{9}\right)^{-3/2}$; (f) $(144)^{-3/2}$.

5. Express each of the following twelve expressions under a common radical:

(a) $\sqrt[3]{a} \cdot \sqrt[3]{b}$; (b) $a\sqrt[3]{a}$; (c) $\sqrt{3} \cdot \sqrt[3]{3}$; (d) $\sqrt{a} \cdot \sqrt[3]{b}$;

(e) $2\sqrt[3]{5}$; (f) $-2\sqrt[5]{3}$; (g) $3\sqrt{2} \cdot 2\sqrt[4]{3}$; (h) $\frac{\sqrt[3]{9}}{\sqrt{3}}$;

(i) $\frac{\sqrt[5]{30}}{\sqrt[5]{5}}$; (j) $\frac{\sqrt[5]{-30}}{\sqrt[5]{-5}}$; (k) $\frac{\sqrt{a}}{\sqrt[3]{a}}$; (l) $\frac{\sqrt[3]{7}}{\sqrt[4]{7}}$.

6. Simplify each of the following twelve expressions:

(a) $\sqrt[3]{4} \cdot \sqrt[3]{2}$; (b) $\sqrt[3]{-4} \cdot \sqrt[3]{-2}$; (c) $\sqrt{64} \cdot \sqrt[3]{64}$;

(d) $\sqrt{-64} \cdot \sqrt[3]{-64}$; (e) $(2\sqrt[3]{2})^3$; (f) $(-\frac{1}{2}\sqrt{3})^2$;

(g) $2\sqrt{\frac{3}{2}} \cdot 3\sqrt{\frac{32}{3}}$; (h) $3 \cdot \sqrt[3]{4} \cdot \sqrt[6]{108}$;

(i) $\sqrt[3]{\frac{3}{4}} \div \sqrt[3]{\frac{2}{45}}$; (j) $3\sqrt[3]{18} \div \frac{3}{2}\sqrt[6]{54}$;

(k) $(\sqrt[4]{6} \div \sqrt[3]{12})\sqrt[6]{24}$; (l) $\sqrt{12ab^2c^3d^0}$.

For each of the following (Exercises 7 through 12) find a value of x which will satisfy the given equation.

7. $x^{\frac{1}{2}} = 3$. **8.** $x^{\frac{3}{2}} = 8$. **9.** $x^{\frac{2}{3}} = \frac{25}{16}$.

10. $x^{-5} = 32$. **11.** $x^{-\frac{3}{2}} = \frac{1}{27}$. **12.** $x^{-\frac{3}{4}} = 8$.

Perform the addition of radicals in Exercises 13 through 28.

13. $\sqrt{12} - \sqrt{75} + 2\sqrt{3}$.

14. $\sqrt{80} - 5\sqrt{20} + 3\sqrt{125} - 4\sqrt{320}$.

15. $3\sqrt{125} - 2\sqrt{363} + 4\sqrt{243} - 2\sqrt{147}$.

16. $\sqrt[3]{16} - 3\sqrt[3]{54} + 2\sqrt[3]{250}$.

17. $\sqrt{45} - \sqrt{150} + 2\sqrt{\frac{9}{5}}$.

18. $3\sqrt{2} - 2\sqrt{3} + 4\sqrt{3} - 5\sqrt{2}$.

19. $\sqrt{\frac{2}{3}} - 3\sqrt{600} + 5\sqrt{864} - \sqrt{\frac{3}{2}}$.

20. $\frac{1}{2}\sqrt{5} - \frac{1}{5}\sqrt{2} - 5\sqrt{2} + 2\sqrt{2}$.

21. $\sqrt{\frac{2}{5}} - 3\sqrt{\frac{9}{10}} + 4\sqrt{\frac{72}{45}}$.

22. $4\sqrt{2} - 3\sqrt{5} + 2\sqrt{2} + 3\sqrt{3} - 4\sqrt{5} - 5\sqrt{2} + \sqrt{2}$.

23. $\sqrt[3]{-3{,}000} + 4\sqrt[3]{-375} + \sqrt[3]{192}$.

24. $\dfrac{\sqrt{\frac{3}{4}} + \sqrt{\frac{4}{3}}}{\sqrt{\frac{3}{4}} - \sqrt{\frac{4}{3}}}$.

25. $3\sqrt{45} - 2\sqrt{20} + 3\sqrt{5} - 4\sqrt{125} + 5\sqrt{80}$.

26. $16\frac{3}{4}\sqrt{\frac{8}{3}} - \frac{2}{3}\sqrt{\frac{27}{8}} + \frac{1}{2}\sqrt{\frac{25}{6}}$.

27. $\sqrt{\dfrac{9}{2x}} - 3\sqrt{\dfrac{1}{18x}}$.

28. $\sqrt{\dfrac{x+1}{x-1}} - \dfrac{1}{x^2-1}\sqrt{4 - \dfrac{4}{x^2}}$.

Simplify the expressions in Exercises 29 through 40.

29. $(\sqrt{2} - 1)^2$. **30.** $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$.

31. $(1 + 2\sqrt{3})^2$. **32.** $\sqrt{2a}\sqrt{10ab}$.

33. $(\sqrt{2} - 1)(\sqrt{2} + 1)$. **34.** $(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})$.

35. $(1 - 2\sqrt{3})(1 + 2\sqrt{3})$. **36.** $(\sqrt{3} + \sqrt{2})^2$.

37. $(-\sqrt{3}-\sqrt{2})(-\sqrt{3}+\sqrt{2})^2$.

38. $(3+2\sqrt{3})(3-2\sqrt{3})^2$.

39. $(\sqrt{x+y}-\sqrt{x-y})^2$.

40. $(\sqrt{x+y}-\sqrt{x-y})(\sqrt{x+y}+\sqrt{x-y})$.

Arrange the expressions in each of the following four exercises in order of magnitude.

41. $\sqrt{2}, \sqrt[3]{3}, \sqrt[5]{5}$.

42. $\sqrt[3]{2}, \sqrt[9]{8}, \sqrt[6]{4}$.

43. $\sqrt{5}, \sqrt[3]{11}, \sqrt[6]{13}$.

44. $\sqrt{2}, \sqrt[4]{3}, \sqrt[6]{9}$.

45. Compare the values of $\sqrt{9+4}$ and $\sqrt{9}+\sqrt{4}$.

46. What is necessary to make $\sqrt{a^2+(\quad)+b^2}$ equal to $a+b$?

47. Find the value of each of the following twelve expressions and check the results (a) through (i) by the use of the table:

(a) $(12)^3$; (b) $(120)^2$; (c) $(0.67)^2$; (d) $(0.67)^3$;
(e) $(97)^2$; (f) $(87)^3$; (g) $\sqrt{324}$; (h) $\sqrt{4{,}500}$;
(i) $\sqrt[3]{62}$; (j) $\sqrt[3]{620}$; (k) $\sqrt[3]{6{,}200}$; (l) $\sqrt{.062}$.

48. Find the value of each of the following twelve expressions and check the results (a) through (i) by the use of the table:

(a) $(190)^2$; (b) $(32)^3$; (c) $\sqrt{5{,}400}$;
(d) $\sqrt[3]{100{,}000}$; (e) $\sqrt{9800}$; (f) $\sqrt{0.0024}$;
(g) $\sqrt[3]{0.00089}$; (h) $\sqrt[3]{0.0001}$; (i) $\sqrt[3]{0.00076}$;
(j) $\sqrt{7{,}396}$; (k) $\sqrt{74{,}529}$; (l) $\sqrt[3]{41{,}421{,}736}$.

23. SIMPLIFICATION OF ALGEBRAIC EXPRESSIONS

By simplifying an algebraic expression involving different powers of the same number or letter combined by **multiplication** and **division,** we mean combining the exponents of the same numbers and the same letters according to the laws of operation for exponents.

Since the laws of operation hold for all rational values of the exponents, whether they are integers or fractions and whether they are positive or negative, the laws of operation, rather than the definitions, should be used whenever possible. Perhaps the most important use of the definitions is to change radicals to fractional exponents before applying the laws of operation, and to change the form of the answer to a required specified form.

The laws of operation are commutative, thus they can be applied in any order. However, usually the work may be simplified by applying Law I before Laws II and III, or by applying Law II before Law III whenever possible. To illustrate,

$$\left(\frac{x^{-2}y^3}{x^3y^{-2}}\right)^{-\frac{1}{5}} = (x^{-5}y^5)^{-\frac{1}{5}} = xy^{-1}, \tag{15}$$

is preferable to

$$\left(\frac{x^{-2}y^3}{x^3y^{-2}}\right)^{-\frac{1}{5}} = \frac{x^{\frac{2}{5}}y^{-\frac{3}{5}}}{x^{-\frac{3}{5}}y^{\frac{2}{5}}} = xy^{-1}. \tag{16}$$

In Equation (15), Law II (division) was used before using Law III (removing parentheses). In Equation 16, Law III (removing parentheses) was used before Law II (division).

If an expression containing a power or a root of a sum or difference is to be simplified, an attempt should be made to represent the expression as a product or quotient before simplifying; then, apply the laws of exponents to each factor. To illustrate,

$$[(a+b)^2(x-y)^{-5}]^3 = (a+b)^6(x-y)^{-15}, \tag{17}$$

and

$$\sqrt{a^6 - a^4b^2} = \sqrt{a^4(a^2-b^2)} = a^2\sqrt{a^2-b^2}. \tag{18}$$

When simplifying expressions containing numbers, it is sometimes helpful to express each number as a product of powers of its different prime factors before simplifying. For example, write 36 as 2^23^2, 48 as 2^43, and so on.

When simplifying an algebraic expression involving multiplication, division, roots, and powers, the result can be expressed in any one of the following three forms:

FIRST FORM. Express the result as a product only, using positive and negative exponents, so that all the different powers of the same numbers or letters are combined into a single power of these same numbers or letters; as:

Example 1.

$$\left(\frac{x^{-2}y^3}{x^3y^{-2}}\right)^{-\frac{1}{5}} = (x^{-5}y^5)^{-\frac{1}{5}} = xy^{-1}.$$

Example 2.

$$\sqrt[3]{\frac{a^7b^2}{16a^2b^4}} = (2^{-4}a^5b^{-2})^{\frac{1}{3}} = 2^{-\frac{4}{3}}a^{\frac{5}{3}}b^{-\frac{2}{3}}.$$

Example 3.

$$\sqrt[6]{48a^2b^3c^{12}} = \sqrt[6]{2^4 \cdot 3a^2b^3c^{12}} = 2^{4/6}3^{1/6}a^{2/6}b^{3/6}c^{12/6} = 2^{2/3}3^{1/6}a^{1/3}b^{1/2}c^2.$$

Example 4.

$$\sqrt[5]{\frac{1}{2^3}} = (2^{-3})^{1/5} = 2^{-3/5}.$$

Example 5.

$$\frac{\sqrt{48x^3y^5}}{\sqrt[3]{4xy^8z^7}} = \frac{(2^4 \cdot 3 \cdot x^3y^5)^{1/2}}{(2^2xy^8z^7)^{1/3}} = \frac{2^2 \cdot 3^{1/2} \cdot x^{3/2}y^{5/2}}{2^{2/3}x^{1/3}y^{8/3}z^{7/3}} = 2^{4/3}3^{1/2}x^{7/6}y^{-1/6}z^{-7/3}.$$

SECOND FORM. Express the result as a quotient, using positive exponents only, so that the different powers of the same numbers or letters are combined into a single power of these same numbers or letters. To illustrate, the foregoing examples may be simplified as follows:

Example 1.

$$\left(\frac{x^{-2}y^3}{x^3y^{-2}}\right)^{-1/5} = \left(\frac{y^5}{x^5}\right)^{-1/5} = \frac{y^{-1}}{x^{-1}} = \frac{x}{y}.$$

Example 2.

$$\sqrt[3]{\frac{a^7b^2}{16a^2b^4}} = \left(\frac{a^5}{2^4b^2}\right)^{1/3} = \frac{a^{5/3}}{2^{4/3}b^{2/3}}.$$

Example 4.

$$\sqrt[5]{\frac{1}{2^3}} = \left(\frac{1}{2^3}\right)^{1/5} = \frac{1}{2^{3/5}}.$$

Example 5.

$$\frac{\sqrt{48x^3y^5}}{\sqrt[3]{4xy^8z^7}} = \frac{(2^4 \cdot 3 \cdot x^3y^5)^{1/2}}{(2^2xy^8z^7)^{1/3}} = \frac{2^2 \cdot 3^{1/2} \cdot x^{3/2}y^{5/2}}{2^{2/3}x^{1/3}y^{8/3}z^{7/3}} = \frac{2^{4/3} \cdot 3^{1/2}x^{7/6}}{y^{1/6}z^{7/3}}.$$

THIRD FORM. Express the result in radical form so that: (a) *the index of the radical is a positive integer;* (b) *there are no fractions under the radical sign;* (c) *there are no radicals in the denominator;* (d) *there are no factors under the radical sign with* (1) *a negative or fractional exponent,* (2) *an exponent greater than the index of the root;* (e) *the exponents of all the factors in the radicand do not have a common factor which is a factor of the index.*

The process of changing a fraction with one or more radicals in the denominator to an equivalent form where the denominator is free of radicals is called ***rationalizing the denominator.***

To rationalize a radical of index **n** *of a fraction, multiply both numerator and denominator of the fraction by an expression that*

will make the denominator a perfect **n***th power; that is, make the exponents of every factor in the denominator a multiple of* **n.** See the following Examples 2, 4, and 5.

To rationalize the denominator of a fraction containing a radical of index **n** *in the denominator, multiply both numerator and denominator of the fraction by a radical having an index* **n** *and a radicand so chosen that the product of this radicand by the radicand in the denominator of the given fraction is a perfect* **n***th power.* See Example 4′.

We shall now simplify according to radicals the foregoing examples containing radicals.

Example 2.

$$\sqrt[3]{\frac{a^7b^2}{16a^2b^4}} = \sqrt[3]{\frac{a^5}{2^4b^2}} = \sqrt[3]{\frac{a^5}{2^4b^2}\cdot\frac{2^2b}{2^2b}} = \sqrt[3]{\frac{a^3 2^2 a^2 b}{2^6b^3}} = \frac{a}{2^2b}\sqrt[3]{2^2a^2b}.$$

Example 3.

$$\sqrt[6]{48a^2b^3c^{12}} = c^2\sqrt[6]{48a^2b^3}.$$

Example 4.

$$\sqrt[5]{\frac{1}{2^3}} = \sqrt[5]{\frac{1}{2^3}\cdot\frac{2^2}{2^2}} = \sqrt[5]{\frac{2^2}{2^5}} = \frac{\sqrt[5]{2^2}}{2} = \frac{1}{2}\sqrt[5]{2^2}.$$

Example 4′.

$$\sqrt[5]{\frac{1}{2^3}} = \frac{1}{\sqrt[5]{2^3}} = \frac{1}{\sqrt[5]{2^3}}\cdot\frac{\sqrt[5]{2^2}}{\sqrt[5]{2^2}} = \frac{\sqrt[5]{2^2}}{2} = \frac{1}{2}\sqrt[5]{2^2}.$$

Example 5.

$$\frac{\sqrt{48x^3y^5}}{\sqrt[3]{4xy^8z^7}} = \frac{\sqrt{2^4 3x^3y^5}}{\sqrt[3]{2^2xy^8z^7}} = \sqrt[6]{\frac{2^{12}3^3x^9y^{15}}{2^4x^2y^{16}z^{14}}} = \sqrt[6]{\frac{2^8 3^3 x^7}{yz^{14}}}$$

$$= \sqrt[6]{\frac{2^8 3^3 x^7}{yz^{14}}\cdot\frac{y^5z^4}{y^5z^4}} = \sqrt[6]{\frac{2^8 3^3 x^7 y^5 z^4}{y^6z^{18}}} = \frac{2x}{yz^3}\sqrt[6]{2^2 3^3 xy^5z^4}.$$

Example 6.

$$\sqrt[9]{x^{12}y^6} = \sqrt[3]{x^4y^2} = x\sqrt[3]{xy^2}.$$

Let us observe the difference in meaning between the sum of powers of numbers and a power of a sum of numbers. For example, observe the difference in the meaning of $a^2 + b^2$ and $(a + b)^2$; $(a^{-1} + b^{-1})$ and $(a + b)^{-1}$; $(a + b)^{-2}$ and $(a^{-2} + b^{-2})$ and so forth.

In the case of a sum or difference of powers, simplify each term of the sum separately, as:

$$a^{-1} + b^{-1} = \frac{1}{a} + \frac{1}{b}, \tag{19}$$

and

$$a^{-2} + b^{-2} = \frac{1}{a^2} + \frac{1}{b^2}, \tag{20}$$

and

$$a^{\frac{1}{2}} - b^{\frac{1}{2}} = \sqrt{a} - \sqrt{b}. \tag{21}$$

In the case of a power of a sum or difference, the sum must be treated as a single quantity as:

$$(a + b)^2 = a^2 + 2ab + b^2, \tag{22}$$

and

$$(a + b)^{-1} = \frac{1}{a + b}, \tag{23}$$

and

$$(a + b)^{-2} = \frac{1}{(a + b)^2}, \tag{24}$$

and

$$(a - b)^{\frac{1}{2}} = \sqrt{a - b}. \tag{25}$$

Example 7. Fractions like $\dfrac{x^{-1} + y^{-1}}{x^{-2} - y^{-2}}$ can be simplified by either of the following two methods.

Solution. *First Method.* Apply definition 3 for negative exponents to all of the terms, in both the numerator and the denominator, that contain negative exponents. Thus,

$$\frac{x^{-1} + y^{-1}}{x^{-2} - y^{-2}} = \frac{\dfrac{1}{x} + \dfrac{1}{y}}{\dfrac{1}{x^2} - \dfrac{1}{y^2}} = \frac{\dfrac{y + x}{xy}}{\dfrac{y^2 - x^2}{x^2y^2}}$$

$$= \frac{y + x}{xy} \cdot \frac{x^2y^2}{y^2 - x^2} = \frac{xy}{y - x}.$$

Solution. *Second Method.* Fractions containing sums and differences of negative powers of numbers and letters can be simplified by multiplying both numerator and denominator by such an expression as will make all the exponents in the new numerator and the new denominator positive. Thus, $\dfrac{x^{-1} + y^{-1}}{x^{-2} - y^{-2}}$ can be simplified by multiplying both numerator and denominator by x^2y^2. Hence,

$$\frac{x^{-1} + y^{-1}}{x^{-2} - y^{-2}} = \frac{xy^2 + x^2y}{y^2 - x^2} = \frac{xy(y + x)}{(y - x)(y + x)} = \frac{xy}{y - x}.$$

Since

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$$
$$= (\sqrt{a})^2 + \sqrt{a}\sqrt{b} - \sqrt{a}\sqrt{b} - (\sqrt{b})^2 = a - b, \tag{26}$$

we see that it is possible to get rid of the radicals in the denominator in the following manner.

To rationalize the denominator of a fraction whose denominator is composed of the sum (or difference) of two terms, one or both of which contains a square root, multiply the numerator and the denominator of the given fraction by the difference (or sum) of the two terms in the denominator.

Example 8. Rationalize the denominator of $\dfrac{\sqrt{3}}{\sqrt{5}+3\sqrt{2}}$.

Solution. Multiplying both numerator and denominator by $\sqrt{5}-3\sqrt{2}$, we have

$$\frac{\sqrt{3}}{\sqrt{5}+3\sqrt{2}}=\frac{\sqrt{3}}{\sqrt{5}+3\sqrt{2}}\cdot\frac{\sqrt{5}-3\sqrt{2}}{\sqrt{5}-3\sqrt{2}}=\frac{\sqrt{15}-3\sqrt{6}}{5-9\cdot 2}$$

$$=\frac{\sqrt{15}-3\sqrt{6}}{-13}=-\frac{1}{13}\sqrt{15}+\frac{3}{13}\sqrt{6}.$$

If a fraction has for its denominator the sum of several radicals, or the sum of radicals with indices greater than 2, the process of simplification becomes very complex and may be impossible.

In a few special cases, we can obtain an alternate expression for radicals, as in the following manner.

$$\sqrt{5+2\sqrt{6}}=\sqrt{2+2\sqrt{6}+3}=(\sqrt{2}+\sqrt{3})^2=\sqrt{2}+\sqrt{3}.$$

When simplifying a radical with a large number for the radicand, it is best to write the radicand as the product of powers of its prime factors. One way to do this is to divide the radicand by one of its prime factors and repeat with the quotient until all the prime factors are found. To illustrate, simplify

$$\sqrt[3]{6{,}480}.$$

The prime factors of 6,480 can be determined as follows:

$$\begin{array}{r|l}
5 & 6{,}480 \\
2 & 1{,}296 \\
2 & 648 \\
2 & 324 \\
2 & 162 \\
3 & 81 \\
3 & 27 \\
3 & 9 \\
 & 3
\end{array}$$

Therefore,

$$\sqrt[3]{6{,}480} = \sqrt[3]{2^4 \cdot 3^4 \cdot 5} = 2 \cdot 3\sqrt[3]{2 \cdot 3 \cdot 5} = 6\sqrt[3]{30}.$$

QUESTIONS

1. What is meant by simplifying an expression involving multiplication, division, and powers of letters and numbers?

2. What is meant by simplifying an expression containing sums and differences of powers?

3. How is the denominator containing one radical rationalized?.

4. How is the denominator consisting of the sum or difference of two square roots rationalized?

EXERCISES

Simplify each of the following (Exercises 1 through 72):

1. $\left(\frac{x^2y^3}{x^3y^2}\right)^{-1}$.

2. $\left(\frac{x^{-3}y^2}{xy^{-5}}\right)^3$.

3. $\left(\frac{x^{-2}y^4}{x^4y^{-2}}\right)^{1/6}$.

4. $\frac{(x^{2/3}y)^4}{\sqrt[3]{x^5y^2}}$.

5. $\left(\frac{x^{-4}y^3}{x^4y^{-3}}\right)^{-1/4}$.

6. $\sqrt[3]{\frac{a^2b^4}{a^5b^3}}$.

7. $\left(\frac{x^{-2}y^4}{x^4y^{-2}}\right)^{1/6}\left(\frac{x^{-4}y^3}{x^4y^{-3}}\right)^{-1/4}$.

8. $(2x^{1/2}y^{-1/6}z^{1/4})^4$.

9. $\frac{a^{3/4}b^{-2/3}}{a^{-1}b^{3/4}}$.

10. $\frac{4x^{-2/3}y^{-3/4}}{5x^{1/6}y^{-7/8}}$.

11. $\left(\frac{xy^{-3/4}}{\sqrt{x}}\right)^{-1/3}$.

12. $\frac{x^{-1}y^{-2}}{x^{-2}y^{-1}}$.

13. $(\sqrt[3]{x^3y^2})^6$.

14. $(x^{3/4}y^{-2/3})^{3/2}$.

15. $\left(\frac{x^{a+b}}{x^a}\right)^b \div \left(\frac{x^{a+b}}{x^b}\right)^a$.

16. $(x^3y^az^{3a})^{1/3a}$.

17. $\frac{2^{n+1}}{(2^n)^{n-1}} \div \frac{4^{n+1}}{(2^{n-1})^{n+1}}$.

18. $(x^ny^nz^{2n})^{1/n}$.

19. $(x^{10a} \cdot x^{-15})^{1/(2a-3)}$.

20. $\sqrt[9]{(x^4y^8)^{3/8}}$.

21. $[(x^a)^{a-\frac{1}{a}}]^{1/(a+1)}$.

22. $(x^{1/(n-1)} \cdot x^{1/(n+1)})^{n+1}$.

23. $\left[x^{4/3}\sqrt[3]{\frac{(x^{3/8})^0}{\sqrt[5]{x^2}}}\right]^{5/6}$.

24. $\left[a^{2/3}\sqrt[3]{\frac{\sqrt[3]{a}}{a^{1/2}}}\right]^{9/4}$.

25. $\frac{3^{n+1}}{(3^n)^{n-1}} \div \frac{9^{n+1}}{(3^{n-1})^{n+1}}$.

26. $\frac{(a^{-2}b)^{-3}}{(a^{-1}b^3)^2} \div \frac{(a^2b^{-3})^{-2}}{(a^{-3}b^2)^{-3}}$.

27. $\left(\frac{a^{-2}b}{a^3b^{-4}}\right)^{-3} \div \left(\frac{ab^{-1}}{a^{-3}b^2}\right)^5$.

28. $\frac{(x^{3/4}y^{-4/3})^{4/3}}{(x^{-4/3}y^{3/4})^{-3/4}}$.

29. $\left(\frac{y^{-2}}{x^{3/5}z^{-3}}\right)^{-2/3}\left(\frac{y^{15/2}x^{-2}}{z^{-10/3}}\right)^{2/5}$.

30. $\left(\frac{x^{-1}y^{-2/3}}{x^{3/4}y^{-3/2}}\right)^{-3/2}\left(\frac{x^{-3/2}y^{-1}}{x^{-2/3}y^{3/4}}\right)^{2/3}$.

31. $\sqrt[3]{a\sqrt{a^3}}$.

32. $\sqrt[3]{x^2\sqrt{x^3}}$.

33. $\sqrt{x\sqrt{x\sqrt{x}}}$.

34. $\sqrt[3]{x^2\sqrt[3]{x^2\sqrt[3]{x^2}}}$.

35. $\sqrt{\frac{64a^7b}{8}\sqrt{\frac{8}{a^7b}}}$.

36. $\sqrt[5]{\sqrt[3]{x^2\sqrt{x\sqrt{x}}}}$.

37. $\sqrt{12}\cdot\sqrt[3]{36}$.

38. $2\sqrt[6]{8y^2}\cdot\sqrt[6]{16y^5}$.

39. $\sqrt{a^{-2}b}\cdot\sqrt[3]{ab^{-3}}$.

40. $\left(\frac{xy^{-\frac{3}{4}}}{\sqrt{xz^0}}\right)^{-\frac{1}{3}}$.

41. $\frac{\sqrt{x^3y^4z^5}}{\sqrt[3]{x^6y^7z^8}}$.

42. $\left(\frac{x^{-3}}{y^{\frac{3}{4}}a^{-1}}\right)^{-3\frac{1}{2}} \div \left(\frac{a^{-2\frac{1}{4}}}{x^2y^{-1}}\right)^{\frac{3}{7}}$.

43. $\frac{1}{\sqrt[4]{\frac{1}{\sqrt[3]{9}}}}$.

44. $\sqrt[3]{\frac{a^5b^{-2}}{9b^7}}$.

45. $\sqrt{x^4y^9z^{25}}$.

46. $\sqrt{16a^2x^3 - 64a^4x^4}$.

47. $\left(\frac{16x^4}{81y^{-4}}\right)^{-\frac{3}{4}}$.

48. $\frac{148xy^{-3}}{8x^{-\frac{2}{3}}y^{\frac{3}{4}}}$.

49. $\sqrt[5]{\frac{x^2y\sqrt{xy^{-1}}z^{-3}x^2z^0}{x^3y^{-1}z^{-3}\sqrt[3]{x}}}$.

50. $\left[\frac{\sqrt{xy^{-7}z^{-3}}x^5z^{-3}\sqrt[3]{xz^7y^5}}{\sqrt[4]{(x^2y)^2}z^{-3}x^2(xy^{-1})^3}\right]^{-2}$.

51. $\sqrt[3]{\frac{54x^4y^2z^3}{24ab^2c^5}}$.

52. $\frac{\sqrt[6]{20x^4}}{\sqrt[3]{10x^2}}$.

53. $\sqrt[6]{4x^9b^8c^{\frac{1}{3}}}$.

54. $\sqrt[4]{3{,}111{,}696x^2y^6}$.

55. $\sqrt[6]{a^8b^{-4}c^9}$.

56. $(\frac{1}{32}y^{10a})^{\frac{1}{5}}$.

57. $\sqrt{53{,}361a^4b^6c^9d^0}$.

58. $(x^{\frac{3}{4}}x^{-\frac{4}{3}})^{-\frac{5}{6}}$.

59. $\sqrt{9{,}000}$.

60. $\sqrt[3]{432}$.

61. $\sqrt{512{,}000}$.

62. $\sqrt[4]{15{,}552}$.

63. $x^{-3} - y^{-3}$.

64. $a^{-\frac{1}{2}} + b^{-\frac{1}{2}}$.

65. $\frac{x^{-3}}{x^{-3} - y^{-3}}$.

66. $\frac{a^{-2} + b^{-2}}{a^2 + b^2}$.

67. $\frac{1}{x^{-3} - y^{-3}}$.

68. $\left(\frac{x^{-1} + y^{-1}}{x^{-2} - y^{-2}}\right)^{-2}$.

69. $(x - y)^{-3}$.

70. $\frac{3}{4 - 3b^{-2}}$.

71. $\frac{1}{1 - x^{-3}}$.

72. $\frac{2^{-1}a^{-2}b^{-3}}{3a^3b^2}$.

Rationalize the denominator in each of the following (Exercises 73 through 94):

73. $\sqrt{\frac{1}{2}}$.

74. $\frac{1}{\sqrt{5}}$.

75. $\sqrt{\frac{3}{10}}$.

76. $\sqrt{\frac{2}{5}}$.

77. $\sqrt{\frac{72}{81}}$.

78. $\sqrt{\frac{1}{8}}$.

79. $\sqrt[3]{\frac{2}{3}}$.

80. $\frac{1}{\sqrt[3]{5}}$.

81. $\sqrt[3]{\frac{3}{4}}$. **82.** $\frac{2\sqrt[3]{9}}{6\sqrt{3}}$. **83.** $\sqrt[3]{\frac{12}{32}}$. **84.** $\sqrt[3]{\frac{125}{384}}$.

85. $\sqrt[3]{\frac{1}{a^2}}$. **86.** $\sqrt[3]{\frac{1}{a}}$. **87.** $\frac{2}{1-\sqrt{2}}$. **88.** $\frac{\sqrt{2}}{1+\sqrt{2}}$.

89. $\frac{\sqrt{3}+4}{\sqrt{3}-4}$. **90.** $\frac{1-\sqrt{2}}{1+\sqrt{2}}$. **91.** $\frac{2\sqrt{2}+3\sqrt{3}}{3\sqrt{3}-2\sqrt{2}}$.

92. $\frac{\sqrt{2}+\sqrt{5}}{\sqrt{2}-\sqrt{5}}$. **93.** $\frac{1}{2+\sqrt{2}+\sqrt{3}}$. **94.** $\frac{1}{1-\sqrt{2}+\sqrt{3}}$.

24. MULTIPLICATION AND DIVISION OF ALGEBRAIC EXPRESSIONS

To multiply two algebraic expressions composed of sums of terms, multiply every term of the one expression by each term of the other expression; then add the coefficients of like terms. For example,

$$2(x^4 - 3x + 1) = 2x^4 - 6x^2 + 2, \tag{27}$$

and

$$a(x^4 - 3x^2 + 1) = ax^4 - 3ax^2 + a, \tag{28}$$

and

$$\begin{aligned}(x^2 - 2x - 1)&(x^4 - 3x^2 + 1)\\ &= x^4(x^2 - 2x - 1) - 3x^2(x^2 - 2x - 1) + \\ &\quad (x^2 - 2x - 1)\\ &= x^6 - 2x^5 - x^4 - 3x^4 + 6x^3 + 3x^2 + x^2 - 2x - 1\\ &= x^6 - 2x^5 - 4x^4 + 6x^3 + 4x^2 - 2x - 1. \end{aligned}\tag{29}$$

For convenience in adding, the work may be arranged as follows. It usually facilitates the work if both the multiplier and multiplicand are arranged in either descending or ascending powers of some common letter.

Example 1. Multiply $(x^2 - 2x - 1)$ by $(x^4 - 3x^2 + 1)$.

Solution.

$$\begin{array}{r} x^2 - 2x - 1 \\ x^4 - 3x^2 + 1 \\ \hline x^6 - 2x^5 - x^4 \\ - 3x^4 + 6x^3 + 3x^2 \\ x^2 - 2x - 1 \\ \hline x^6 - 2x^5 - 4x^4 + 6x^3 + 4x^2 - 2x - 1 \end{array}$$

It is always possible to check the product of two algebraic expressions by assigning values other than 1 or 0 to each of the letters in the multiplier, the multiplicand, and the product.

To illustrate, let $x = 2$ in Example 1, then

$$(x^2 - 2x - 1)(x^4 - 3x^2 + 1) = x^6 - 2x^5 - 4x^4 + 6x^3 + 4x^2 - 2x - 1$$

becomes

$$(4 - 4 - 1)(16 - 12 + 1) = 64 - 64 - 64 + 48 + 16 - 4 - 1$$

or

$$(-1)(5) = -5,$$

which indicates that the multiplication is correct.

To divide two algebraic expressions composed of sums of terms, it is highly recommended that both dividend and divisor be arranged in ***descending*** *powers of some common letter before dividing.**

Divide the first term of the dividend by the first term of the divisor for the first term of the quotient. Then multiply all the terms of the divisor by this term of the quotient and subtract the product from the dividend. Repeat for the remaining terms of the quotient until the remainder is either zero, or of lower ***degree*** *in this common letter than the divisor.*

Example 2. Divide $5x - 3x^2 + 2x^4$ by $x^2 + 3 - 2x$.

Solution. After arranging the terms in descending powers of x, we have

$$\begin{array}{rl}
2x^4 - 3x^2 + 5x & \big| \; x^2 - 2x + 3 \\
\cline{2-2}
2x^4 - 4x^3 + 6x^2 & \big| \; 2x^2 + 4x - 1 \\
\cline{1-1}
4x^3 - 9x^2 + 5x & \\
4x^3 - 8x^2 + 12x & \\
\cline{1-1}
-x^2 - 7x & \\
-x^2 + 2x - 3 & \\
\cline{1-1}
-9x + 3 &
\end{array}$$

Hence, the quotient is $2x^2 + 4x - 1$ and the remainder is $-9x + 3$. We can now write Example 2 as

$$\frac{2x^4 - 3x^2 + 5x}{x^2 - 2x + 3} = 2x^2 + 4x - 1 + \frac{-9x + 3}{x^2 - 2x + 3},$$

or, since

$$\textbf{dividend} = (\textbf{divisor})(\textbf{quotient}) + \textbf{remainder}, \tag{30}$$

we can write Example 2 as

$$2x^4 - 3x^2 + 5x = (x^2 - 2x + 3)(2x^2 + 4x - 1) + (-9x + 3).$$

* In case there is no remainder, then both dividend and divisor can be arranged in either ascending or descending powers of some common letter.

Let us check these results by letting $x = 2$; then,

$$32 - 12 + 10 = (4 - 4 + 3)(8 + 8 - 1) + (-18 + 3),$$

or

$$30 = (3)(15) - 15,$$

which indicates that the results are correct.

Caution: When checking division by assigning values to the letters it is best to express the result in the form given in Equation (30), and not to assign values to the letters that will make the divisor zero.

Example 3. Divide $x^3 - 3xyz + y^3 + z^3$ by $z + y + x$.

Solution. After arranging the terms in descending powers of x, we have

$$\begin{array}{l|l} x^3 - (3yz)x + (y^3 + z^3) & x + (y + z) \\ \underline{x^3 + (y + z)x^2} & \underline{x^2 - (y+z)x + (y^2 - yz + z^2)} \\ -(y + z)x^2 - (3yz)x & \\ \underline{-(y + z)x^2 - (y + z)^2x} & \\ (y^2 - yz + z^2)x + (y^3 + z^3) & \\ \underline{(y^2 - yz + z^2)x + (y^3 + z^3)} & \\ 0 & \end{array}$$

since $(y + z)^2 = y^2 + 2yz + z^2$ and $(y + z)(y^2 - yz + z^2) = y^3 + z^3$.

In the case of the quotient of two expressions containing two or more letters, the remainder not being zero, it is possible to obtain the result in different forms by using different arrangements of the letters. To illustrate, we can divide $x^2 + y^2$ by $x + y$ if we arrange both the dividend and divisor in descending powers of x, or if we arrange both in descending powers of y. Thus,

$$\begin{array}{l|l} x^2 + y^2 & x + y \\ \underline{x^2 + xy} & \underline{x - y} \\ -xy + y^2 & \\ \underline{-xy - y^2} & \\ 2y^2 & \end{array} \qquad \begin{array}{l|l} y^2 + x^2 & y + x \\ \underline{y^2 + yx} & \underline{y - x} \\ -yx + x^2 & \\ \underline{-yx - x^2} & \\ 2x^2 & \end{array}$$

Both results are correct because

$$\frac{x^2 + y^2}{x + y} = x - y + \frac{2y^2}{x + y} \quad \text{or} \quad y - x + \frac{2x^2}{x + y}.$$

EXERCISES

1. Arrange each of the following five expressions in descending powers of x:

(a) $6x^5 - 4x + 5x^3 + 2 - x^4$;
(b) $x^3 - x^{\frac{1}{2}} + 5x - 2x^{\frac{3}{2}} + 7$;
(c) $3x^{-3} - 4x^4 - 5x^{-2} + x^3 - 2x^5 + 5$;
(d) $5x^2 - 3x + 2 - ax^2 + bx - c$;
(e) $2x^3 - 3ax^2 + ax^3 - 5x^2 - b^3 + 7 - ax + 2x + b$.

2. Arrange each of the following five expressions in descending powers x and of y:

(a) $3x^2y^2 - 2x^2y + 3xy^2 + 5xy - 4x - 2y + 7$;
(b) $4x^3y - 3xy^3 + 2x^2y^2 - 7xy^2 - 5x^2y - 3x + 2xy - y^2 - 8$;
(c) $2x^4y^2 - 5x^3 - 2x^2y^4 - 3x^3y^3 - 6x^2y + 7xy - 12y + 8xy^2 - 9$;
(d) $6x^2y^4 - 2x^3y^2 - 5x^4y^4 - 2xy^4 - 5x^2y^2 + 7 - 12xy^2 - 2x^3y^4$;
(e) $3a^3 - 4x^2ay^2 + 6xy^3 - 4a^2x^3 - 7x^2y - 5axy - 3x^2a^2y^2 + 4x^2y^2 - 3x^2a^2 + 5a^2y^2$.

In Exercises 3 through 41, perform the indicated operations. (The results may be checked by assigning values to the letters.)

3. $(2x - 5)(3x + 2)$.

4. $(2x - 5y)(3x + 2y)$.

5. $(x + y + z)(x - y + z)$.

6. Multiply $3x^3 - 2x^2 + 1$ by $x^2 - 5$.

7. $(3x^2 - 2xy + y^2)(x^2 - y^2 + 2x - 3y)$.

8. Multiply $2x^2 - xy + y^2 - 7x + 2y - 3$ by $2x - 3y + 2$.

9. Multiply $\frac{1}{3}a + \frac{1}{2}b - \frac{2}{3}c$ by $\frac{3}{2}a - \frac{2}{3}b + \frac{1}{3}c$.

10. Multiply $x^{3n+2} - x^{2n+1} + x^n - x^{-1}$ by $x^{2n+3} - 2x^{n+2} - 3x$.

11. Multiply $x^{\frac{3}{4}} + x^{\frac{1}{4}} - x^{\frac{1}{2}} - 1$ by $x^{\frac{3}{4}} - x^{\frac{1}{2}}$.

12. Multiply $x^{\frac{9}{4}} - 2x^{\frac{3}{2}} + 2x^{\frac{3}{4}} - 3$ by $x^{\frac{5}{4}} - 2x^{\frac{1}{2}} + x^{-\frac{1}{4}}$.

13. Multiply $5a^3 - 3ab^2 + 2a^2b - 4b^3$ by $2a^3 - 3a^2b - 2b^3$.

14. Multiply $2a^3 - 3a^2b + 4b^3$ by $3a^3 - 5a^2b - 2b^3$.

15. Multiply $a^{\frac{6}{5}} - 3a^{\frac{4}{5}} + 2a^{\frac{2}{5}} - 3$ by $a^{\frac{2}{5}} - 2 + 3a^{-\frac{2}{5}}$.

16. Multiply $x^3 - x^{\frac{7}{3}}y^{\frac{3}{2}} + x^{\frac{5}{3}}y^3 - xy^{\frac{9}{2}}$ by $x^3 + x^{\frac{7}{3}}y^{\frac{3}{2}} - xy^{\frac{9}{2}}$.

17. Divide $x^4 - 3x^3 - x^2 + 5x + 2$ by $x - 2$.

18. Divide $6x^3 - 3x^2 - 5x - 6$ by $2x - 3$.

19. Divide $6x^4 - 7x^3 - 2x^2 + x - 10$ by $2 - x + 2x^2$.

20. Divide $4x^5 - 3x^3 + 4x - 2$ by $2x^2 - x + 1$.

21. Divide $x^5 - y^5$ by $x - y$.

22. Divide $x^5 + y^5$ by $x + y$.

23. Divide $x^5 - x^4y - y^5 - 3x^2y^3$ by $x^2 + 2xy + y^2$.

24. Divide $x^4 + x^2y^2 + y^4$ by $x^2 - xy + y^2$.

25. Divide $3x^5 - 2x^4 - 8x^2 + 5x$ by $2x^2 - 3x + 1$.

26. Divide $12y^{5n-3} - y^{4n-2} - 20y^{3n-1} + 19y^{2n} - 10y^{n+1}$ by $4y^{2n} - 3y^{n+1} + 2y^2$.

27. Divide $x^5 + x^4y - x^2y - 2x^3y - x^4 + xy - x^2y^2 + y^2$ by $x^3 + x^2y - y$.

28. Divide $x^2 - 2x^{3/2}y^{1/2} + 3xy - 2x^{1/2}y^{3/2} + y^2$ by $x - x^{1/2}y^{1/2} + y$.

29. Divide $x^{3/4} + x^{1/4} - x^{1/2} - 1$ by $x^{1/4} - 1$.

30. Divide $a^2 - 2a^{5/3} + 5a^{4/3} - 8a + 9a^{2/3} - 6a^{1/3} + 2$ by $a^{2/3} - a^{1/3} + 1$.

31. Divide $2x - 1$ by $3x + 1$.

32. Divide $3x^2 + 8x - 10$ by $5x^2 + 2x - 3$.

33. Divide $x^4y - x^3y^2 - 2x^2yz - x^3y + x^2y - xz^2 - y^2 - yz + x^3z$ by $x^2 - xy - y - z$.

34. Divide $x^3 + 8y^3 + z^3 - 6xyz$ by $x^2 + 4y^2 + z^2 - xz - 2xy - 2yz$.

35. Divide $2x^4 + x^3 + 2x^2 + 1$ by $3x^2 + x + 2$.

36. Divide $x^2 - 4y^2 - 25z^2 + 20yz$ by $x + 2y - 5z$.

37. Divide $x^4yz - x^3y^3 - x^2y^4 + x^2y^3z + xy^4z + xyz^4 - y^4z^2 - x^3z^3 + x^2y^2z^2 - x^2z^4$ by $x^2z - xy^2 + xz^2 - y^3$.

38. Write the quotient of $\dfrac{x^n - y^n}{x - y}$.

39. Write the quotient of $\dfrac{x^n - y^n}{x + y}$ (n even). What is the remainder if n is odd?

40. Write the quotient of $\dfrac{x^n + y^n}{x + y}$ (n odd). What is the remainder if n is even?

41. Write the quotient and remainder of $\dfrac{x^n + y^n}{x - y}$.

CHAPTER IV

Factoring and Fractions

TYPES OF FACTORING

1. $ax + ay = a(x + y)$.
2. $ax + ay + bx + by = a(x + y) + b(x + y) = (x + y)(a + b)$.
3. $x^2 - y^2 = (x + y)(x - y)$.
4. $x^2 + 2xy + y^2 = (x + y)^2$.
5. $x^2 - 2xy + y^2 = (x - y)^2$.
6. $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.
7. $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.
8. $x^3 + 3x^2y + 3xy^2 + y^3 = (x + y)^3$.
9. $x^3 - 3x^2y + 3xy^2 - y^3 = (x - y)^3$.
10. $x^2 + (a + b)x + ab = (x + a)(x + b)$.
11. $x^2 + (a + b)xy + aby^2 = (x + ay)(x + by)$.
12. $acx^2 + (ad + bc)xy + bdy^2 = (ax + by)(cx + dy)$.
13. $x^2 + 2xy + y^2 - z^2 = (x + y)^2 - z^2 = (x + y - z)(x + y + z)$.
14. $x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2$
$= (x^2 + y^2 - xy)(x^2 + y^2 + xy)$.
15. $x^2 + y^2 + z^2 - 2xy + 2xz - 2yz = (x - y + z)^2$.

25. POLYNOMIALS

A function of x of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n \quad (1)$$

(not more than $n + 1$ terms), where n is a **positive integer** or **zero** and the a's are constants, real or imaginary, is called a **rational integral function** of x or a **polynomial** in x. If $a_0 \neq 0$ (a_0 not equal to zero) the function or polynomial is said to be of degree n. Such expressions as

$$2x^3 - 3x^2 + 4x - 7, \quad 6x^2 - 5, \quad \text{and} \quad \sqrt{5}\,x^{10} + \tfrac{3}{4}x^9 - \sqrt{-5}\,x^3 + 2 \quad (2)$$

are examples of rational integral functions of x or polynomials in x. *For an expression to be a polynomial in* **x**, *or a rational integral function in* **x**, *all the exponents of the* **x**'*s must be* ***positive integers,*** *while the coefficients may be any constants, real or imaginary.*

It is possible to have rational integral functions of several variables or polynomials in several variables. To illustrate,

$$6x^2y - 3xyz^3 + 7xy^2$$

is a rational integral function of x, y, z or a polynomial in x, y, z.

A polynomial in the variables x, y, z, $\cdots$ is defined to be the algebraic sum of terms of the type

$$kx^ay^bz^c \cdots,$$

where a, b, c, $\cdots$, the exponents of x, y, z, $\cdots$, are **nonnegative integers** and the coefficient k is any constant, real or imaginary.

The degree of a term of a polynomial with reference to specified variables is the sum of the exponents of these variables in this term. To illustrate, the term

$$6x^3y^4z$$

is of the third degree in x, fourth degree in y, first degree in z, seventh degree in x and y, fourth degree in x and z, fifth degree in y and z, and eighth degree in x, y, and z.

By the degree of a polynomial in specified variables is meant the degree of the term or terms of largest degree in these specified variables. To illustrate, the polynomial

$$3x^5 + x^2y^2z^4 - 4xy^2z^3 + z^7$$

is of the fifth degree in x, second degree in y, seventh degree in z, eighth degree in x, y, and z, and so on.

Any expression that can be represented as the ratio of two polynomials is called a **rational fractional function,** where the polynomial in the denominator is not a constant, as:

$$\frac{x+y}{x-y}, \qquad x^{-1}+5, \qquad \text{and} \qquad x^2+\frac{x}{y}-y^2+\frac{x-y}{x^2+2}.$$

Algebraic functions which are not rational integral functions or rational fractional functions are called **irrational functions,** as:

$$5\sqrt{x}+3 \qquad \text{and} \qquad 8x^{3/2}+7x^{1/2}-2x^3-2.$$

26. FACTORING

If a polynomial can be written as the product of two or more polynomials, then each of the latter polynomials is said to be a **factor** of the given polynomial. Thus, 3, x, and $(x - y)$ are each factors of $3x^2 - 3xy$, which can be written as $3x(x - y)$.

The factors of $x^2 - 5$ are $x + \sqrt{5}$ and $x - \sqrt{5}$. These factors are polynomials in x. However, if $x - y$ is written as $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$, these quantities are not considered as factors of $x - y$ since $\sqrt{x} + \sqrt{y}$ and $\sqrt{x} - \sqrt{y}$ are not rational integral functions in the two letters x and y.

The term factoring is restricted to polynomials. If irrational numbers and imaginary numbers are permitted in the factors, then every polynomial

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$$

can be written as the product of n linear factors as follows (see Equation 16, Chapter XII):

$$a_0(x - r_1)(x - r_2) \cdots (x - r_n).$$

However, in this chapter we shall restrict the term **factor to mean a polynomial with integral coefficients.** Thus in this chapter the factors of $6x^2 + 7x - 5$ are $(2x - 1)$ and $(3x + 5)$. (In Chapter XII it will be convenient to consider $(x + \frac{1}{2})$ and $(x + \frac{5}{3})$ as factors of $6x^2 + 7x - 5$.)

By factoring, in this chapter, we mean writing a given polynomial with integral coefficients as a **product** *of its prime factors, or as powers of prime factors, if some of the prime factors are repeated.* A factor is a prime factor if it cannot be expressed as the product of two polynomials with integral coefficients other than unity and itself or the product of their negatives.

Sometimes the methods that are used to factor polynomials can be used to write an expresssion that is not a polynomial as a product. To illustrate, since $x^2 - y^2 = (x + y)(x - y)$, then $e^{2x} - e^{-2x}$ can be written as $(e^x)^2 - (e^{-x})^2 = (e^x + e^{-x})(e^x - e^{-x})$.

Most of the fundamental types of factoring used are listed at the beginning of this chapter; a complete understanding of them by the student will aid in factoring. As an exercise, the student should verify each type by carrying out the indicated multiplication.

Some of the simpler types of factoring, such as Types 1, 2, 4, and 5, were known to Euclid three centuries before Christ. However, it was centuries later before other factoring types were known, including the factors of $x^2 - y^2$.

A discussion of each type of factoring follows.

TYPE 1. *Common Monomial Factor.*

An example is

$$6x^2 - 12xy^2 = 2 \cdot 3x(x - 2y^2).$$

TYPE 2. *Factoring by Grouping.*

An example is

$$\begin{aligned} 3ax + by - ay - 3bx &= 3ax - 3bx - ay + by \\ &= 3x(a - b) - y(a - b) \\ &= (a - b)(3x - y). \end{aligned}$$

TYPE 3. *Difference of Two Squares.*

The difference of the squares of two numbers is equal to the product of the sum of the two numbers and the first number minus the second number. Thus:

$$\begin{aligned} 16a^2x^4 - 25y^6 &= (4ax^2)^2 - (5y^3)^2 \\ &= (4ax^2 + 5y^3)(4ax^2 - 5y^3). \end{aligned}$$

TYPES 4 AND 5. *Square of a Binomial.*

The square of the sum of two numbers is equal to the square of the first number, plus twice the product of the two numbers, plus the square of the second number [in Type 5 the second number is $-y$].

For example,

$$9x^2 - 12xy^2 + 4y^4 = (3x - 2y^2)^2.$$

Observe that

$$-12xy^2 = 2(3x)(-2y^2).$$

TYPES 6 AND 7. *Sum or Difference of Two Cubes.*

The sum (difference) of the cubes of two numbers is equal to the product of the sum (difference) of the two numbers and the square of the first number minus (plus) the product of the two numbers plus the square of the second number. As an illustration:

$$\begin{aligned} 27x^6 - 8y^3z^9 &= (3x^2)^3 - (2yz^3)^3 \\ &= (3x^2 - 2yz^3)(9x^4 + 6x^2yz^3 + 4y^2z^6). \end{aligned}$$

TYPE 8. *Cube of a Binomial.*

The cube of a sum of two numbers is equal to the cube of the first number, plus three times the product of the square of the first number

and the second number, plus three times the product of the first number and the square of the second number, plus the cube of the second number. For example, if we factor

$$x^3 + 6x^2y^2 + 12xy^4 + 8y^6,$$

we observe that $x^3 = (x)^3$ and $8y^6 = (2y^2)^3$, thus suggesting the following:

$$\begin{aligned} x^3 + 6x^2y^2 + 12xy^4 + 8y^6 &= (x)^3 + 3(x)^2(2y^2) + 3(x)(2y^2)^2 + (2y)^3 \\ &= (x + 2y^2)^3. \end{aligned}$$

TYPE 9. *Cube of a Binomial.*

Observe that the signs alternate in the formula for the cube of a difference.

TYPE 10. *Factoring a Trinomial of the Form*

$$x^2 + (a + b)x + ab = (x + a)(x + b).$$

This type of factoring consists of finding a pair of numbers whose product is the constant term and whose sum is the coefficient of x. An example is

$$x^2 - 5x - 66 = (x + 6)(x - 11).$$

Here $ab = (6)(-11) = -66$, and $a + b = 6 - 11 = -5$.

TYPE 11. This is very similar to Type 10.

TYPE 12. *Factoring a Trinomial of the Form*

$$acx^2 + (ad + bc)xy + bdy^2 = (ax + by)(cx + dy).$$

This type of factoring is accomplished by using a trial and error method. Examine the products of different possible factors until the correct factors are found. As an example, factor

$$6x^2 - 31x + 35.$$

If this example has any factors they must be one of the following forms (since the factors of 6 are either 6 and 1 or 3 and 2):

$$(6x \quad)(x \quad) \qquad \text{or} \qquad (3x \quad)(2x \quad).$$

These take care of the possible choices for a and c. Next, consider all the possible choices for b and d. These are ± 1 and ± 35, or ± 5 and ± 7. Thus, if $6x^2 - 31x + 35$ has any factors they must be included in the following list of products:

$(6x \quad 1)(x \quad 35)$, $(3x \quad 1)(2x \quad 35)$,
$(6x \quad 35)(x \quad 1)$, $(3x \quad 35)(2x \quad 1)$,
$(6x \quad 5)(x \quad 7)$, $(\mathbf{3x \quad 5})(\mathbf{2x \quad 7})$,
$(6x \quad 7)(x \quad 5)$, $(3x \quad 7)(2x \quad 5)$.

Test each of these products to see which is such that $ad + bc = -31$. It is evident that $\pm 7 \cdot 3$ and $\pm 5 \cdot 2$ can be combined to give -31, provided we choose -5 and -7. Therefore,

$$6x^2 - 31x + 35 = (3x - 5)(2x - 7).$$

It should be observed that this is the only combination of the products that can be used.

TYPE 13. This type is reducible to the difference of the squares of two numbers in which one of the numbers is composed of a sum or difference. For example:

$$\begin{aligned} x^2 - y^2 + 4yz - 4z^2 &= x^2 - (y^2 - 4yz + 4z^2) \\ &= x^2 - (y - 2z)^2 \\ &= [x^2 + (y - 2z)][x^2 - (y - 2z)] \\ &= (x^2 + y - 2z)(x^2 - y + 2z). \end{aligned}$$

TYPE 14. This type is also reducible to the difference of the squares of two numbers, usually accomplished by adding and then subtracting the same number which is itself a perfect square. To illustrate:

$$\begin{aligned} 4x^4 - 5x^2y^4 + y^8 &= (4x^4 - 4x^2y^4 + y^8) - x^2y^4 \\ &= (2x^2 - y^4)^2 - (xy^2)^2 \\ &= (2x^2 + xy^2 - y^4)(2x^2 - xy^2 - y^4). \end{aligned}$$

TYPE 15. *Square of a Polynomial.*

The square of the sum of three or more numbers is equal to the sum of the squares of each of the numbers, plus twice all the possible products of the numbers taken two at a time. As an example, factor

$$x^2 + 4y^2 + z^2 - 4xy + 2xz - 4yz.$$

Since x^2, $4y^2$, and z^2 are each a perfect square, it suggests the form $(x \quad 2y \quad z)^2$. Since the terms in y are negative, and the term in xz is positive, it suggests that the signs of the x and z terms are alike and the sign of the y term is opposite to the sign of x and z. Therefore,

$$\begin{aligned} &x^2 + 4y^2 + z^2 - 4xy + 2xz - 4yz \\ &\quad = (x - 2y + z)^2 \qquad \text{or} \qquad (-x + 2y - z)^2. \end{aligned}$$

This can be verified readily by squaring either $(x - 2y + z)$ or $(-x + 2y - z)$.

QUESTIONS

1. Define factor.
2. What is meant by factoring?
3. Give fifteen types of factoring.

EXERCISES

1. Give the degree of the polynomial $x^3 - 6x^4 + 7x^2 - 2$.

2. Give the degree of the polynomial $4x^2y - 2yz^3 + 5x^2yz^4 - 7$ in x; in y; in z; in x and y; in x and z; in y and z; in x, y, and z.

3. Give the degree of each of the following equations in x and y:

(a) $6x^5 - 2x^3 + 1 = 0$; (b) $x^2 + y^2 = 16$;
(c) $y^2 = 3x$; (d) $a^2x^2 + b^2y^2 = a^2b^2$.

4. Give the degree of each of the following equations in x and y:

(a) $3x - 2y + 4 = 0$; (b) $xy = 5$;
(c) $5x^2 - 7xy + 4y^2 = 10$; (d) $y = 3x^2 - 2x + 4$.

In Exercises 5 through 24, perform the indicated operations.

5. $(2x - 3y)(3x - 2y)$.
6. $(x^2 - y^2)^2$.
7. $(a^2 - 3b)(a + 3b^2)$.
8. $(x - 3)(2 - x)$.
9. $(ax + 1)(x - a)$.
10. $(x + y - 3)^2$.
11. $(x - 2y + 3z)^2$.
12. $\left(\frac{1}{2}x - \frac{2}{3}y\right)^2$.
13. $(x + 2y - 3z)^2$.
14. $\left(\frac{2x}{3} - \frac{3y}{2}\right)\left(\frac{2x}{3} + \frac{3y}{2}\right)$.
15. $(3x^2 - 4y)(5x - 6y^2)$.
16. $(x + y + z)(x + y - z)$.
17. $(e^x + e^{-x})^2$.
18. $(e^x - e^{-x})^2$.
19. $(x - 3)(x^2 + 3x + 9)$.
20. $(2a + 3b)(4a^2 - 6ab + 9b^2)$.
21. $(x + y + z)(x - y + z)$.
22. $(2x - 3y)(4x^2 + 6xy + 9y^2)$.
23. $(2\sqrt{x} + 3)(3\sqrt{x} - 5)$.
24. $(e^x + e^{-x})(e^x - e^{-x})$.

Factor each of the following (Exercises 25 through 72):

25. (a) $x^4 - x^3 + x^2 - x$; (b) $x^2 - y^2 - ay + ax$.
26. (a) $3x^3 + 1 - x^2 - 3x$; (b) $x^3 - x^2 + x - 1$.
27. (a) $x^2 - 16$; (b) $x^4 - 16$.
28. (a) $x^4 - y^4$; (b) $x^2 + 4y^2 - 4a^2 - b^2 + 4xy + 4ab$.
29. (a) $x^2 - 10xy + 25y^2$; (b) $x^4 + 10x^2y^2 + 25y^4$.

30. (a) $x^2 - 4xy + 4y^2$; (b) $(x - y)^2 - 2x + 2y + 1$.

31. (a) $x^3 - 27$; (b) $8x^3 + 27y^6$.

32. (a) $x^6 - y^6$; (b) $a^{12} - 64b^6$.

33. (a) $x^3 + 3x^2 + 3x + 1$; (b) $x^3 - 6x^2 + 12x - 8$.

34. (a) $8x^3 + 36x^2y + 54xy^2 + 27y^3$; (b) $x^3 - y^3 - 3x^2y + 3xy^2$.

35. (a) $a^2 - 5a - 84$; (b) $1 + x - 6x^2$.

36. (a) $x^2 - 2x - 15$; (b) $(x^2 - 3x)^2 + (x^2 - 3x) - 12$.

37. (a) $6x^2 - 5xy - 6y^2$; (b) $2x^2 + 5x - 12$.

38. (a) $4x^2 - 11x + 6$; (b) $3x^2y^2 - 9xy^3 - 12y^4$.

39. (a) $x^2 - 2xy + y^2 - z^2$; (b) $x^2 + 2yz - y^2 - z^2$.

40. (a) $b^2 - 4bc + 4c^2 - 9a^2$; (b) $x^4 + 4y^4$.

41. (a) $x^4 + x^2 + 1$; (b) $4x^4 + 3x^2y^2 + y^4$.

42. (a) $25x^4 - 19x^2y^2 + y^4$; (b) $x^4 - 13x^2y^2 + 4y^4$.

43. (a) $x^2 - 2xy + y^2 + 2x - 2y + 1$;
(b) $x^2 + 2xy + y^2 - 4x - 4y + 4$.

44. (a) $x^2 + 4y^2 + z^4 - 4xy + 2xz^2 - 4yz^2$;
(b) $x^2 - 4yz + z^2 + 4xy + 4y^2 - 2xz$.

45. (a) $ax^2 + x - a^2x - a$;
(b) $ax^2 - a^2x + a - x$;
(c) $ax^2 - a^2 + x - a^3x$.

46. (a) $abx^2 + ab - b^2x - a^2x$;
(b) $3a^2 + 3ab - ac - bc$;
(c) $x^2 - 3ax + 6ab - 2bx$.

47. $16x^4 + 4a^2b^4x^2 + a^4b^8$.

48. $x^6y^3 + 3x^4y^2 + 3x^2y + 1$.

49. $6x^2 + 5xy^2 - 6y^4$.

50. $a^3x^6 - 8a^6$.

51. $48x^4 - 243x^2y^4$.

52. $x^8 - 16y^2$.

53. $x^2y^2 - xyz - 90z^2$.

54. $a^3 + 5a^2b + 6ab^2$.

55. $6x^2y^2 + 7xyz - 20z^2$.

56. $x^6y^2 - x^2y^6$.

57. $x^4y^2 + 6x^2yz^2 + 5z^4$.

58. $4x^4 + 8x^2y^2 + 9y^4$.

59. $a(x - y) - b(y - x)$.

60. $2x^2 - 6y^2 + xy + 4xz + 8yz$.

61. $4x^2 - 4y^2 - 12xz + 9z^2$.

62. $6x^3y - 15xy^3 - x^2y^2$.

63. $x^2 - 4ax + 3a^2 - xy + ay$.

64. $6x^4y^2 + 19x^2yab^2 - 36a^2b^4$.

65. $3x^6 - 375a^3b^6$.

66. $4x^4 + a^4y^8$.

67. $a^3 - b^3 + 3b^2x - 3bc^2 + c^3$.

68. $a^3 - b^3 - 3a^2 + 3a - 1$.

69. $4x^2 + 4xy^2 - 15y^4$.

70. $2x^2 + 4xy + 2y^2 + 2ax + 2ay$.

71. $x^3 - 6x^2y + 12xy^2 - 8y^3$.

72. $25x^4 - 60x^2y + 36y^2$.

73. Show why $x^4 + x^2 + 1$ can be factored and $x^4 - x^2 + 1$ cannot be factored.

74. Show why $a^4 - 8a^2b^2 + 4b^4$ can be factored and $a^4 + 8a^2b^2 + 4b^4$ cannot be factored.

Write each of the following expressions (Exercises 75 through 92) as a product.

75. $x - x^{\frac{1}{2}} - 6$.

76. $x^{\frac{1}{2}} + x^{\frac{1}{4}} - 6$.

77. $x^5 + 2x^{\frac{5}{2}} - 8$.

78. $x^{\frac{3}{4}} + 6x^{\frac{3}{2}} - 12$.

79. $\dfrac{x^2}{y^2} + 2 + \dfrac{y^2}{x^2}$.

80. $\dfrac{4x^2}{9y^2} - \dfrac{4x}{3y} + 1$.

81. $x^{-2} + 6x^{-1} + 9$.

82. $x^{-2} + 2(xy)^{-1} + y^{-2}$.

83. $e^{2x} + 2 + e^{-2x}$.

84. $e^{2x} - e^{-2x}$.

85. $x + 2 - 2\sqrt{x+2} + 1$.

86. $x^2 + 5 + 4\sqrt{x^2+1}$.

87. $\dfrac{c}{b^2} + \dfrac{2c}{ab} + \dfrac{c}{a^2}$.

88. $\dfrac{x}{y} + \dfrac{y}{x} - 2$.

89. $x^2 - \dfrac{6}{x^2} + 1$.

90. $3 - \dfrac{6y}{x^2} + \dfrac{3y^2}{x^4}$.

91. $2(x^{\frac{1}{2}} - 1)^2 - x^{\frac{1}{2}} + 1$.

92. $(x^{\frac{1}{3}} - 1)^2 + x^{\frac{2}{3}} - x^{\frac{1}{3}}$.

27. HIGHEST COMMON FACTOR AND LEAST COMMON MULTIPLE

Any polynomial with integral coefficients that is a factor of each polynomial of a set of two or more polynomials with integral coefficients is called a **common factor** or a **common divisor** of these polynomials. To illustrate, 2, x^2, and $x - y$ are common factors of the following set of polynomials which are written in factored form.

$$\left.\begin{aligned} &6x^3(x - y)(x + y), \\ &2x^2(y - x)(x^2 + y^2), \\ &4x^4(x - y)^2(x + y)^2. \end{aligned}\right\} \tag{3}$$

A factor is taken without regard to sign. Thus, either $x - y$ or $y - x$, but not both, is to be considered as a common factor of the foregoing set.

The highest common factor (H.C.F.) of two or more polynomials with integral coefficients is the polynomial of highest degree and with the largest integral coefficients that is a factor of each of the given polynomials.

Any polynomial that contains another polynomial as a factor is said to be a **multiple** of the latter polynomial. Any polynomial that contains each polynomial of a set of two or more polynomials as a factor is said to be a **common multiple** of the set of polynomials.

To illustrate, $(x - y)^2(x + y)^2$ is the common multiple of the following set of polynomials

$$(x^2 - y^2), \qquad (x - y)^2, \qquad (x + y)^2.$$

All polynomials are written in factored form.

The least common multiple (L.C.M.) of two or more polynomials with integral coefficients is the polynomial of lowest degree and with the smallest integral coefficients that has each of the given polynomials as a factor.

The H.C.F. and the L.C.M. of a set of polynomials are most conveniently found when it is possible to factor each polynomial of the set into its prime factors; for then the H.C.F. is the product of all the common prime factors, each factor having for its exponent the **smallest exponent** this factor has in any of the given polynomials, and the L.C.M. is the product of all the different prime factors, each prime factor having for its exponent the **greatest exponent** this factor has in any of the given polynomials. To illustrate:

Example. Find the highest common factor and the least common multiple of $2 \cdot 3x^3(x - y)(x + y)$, $2x^2(y - x)(x^2 + y^2)$, and $2^2x^4(x - y)^2(x + y)^2$.

Solution. The common factors of the set are 2, x^2, and $(x - y)$. Hence, the H.C.F. is $2x^2(x - y)$.

The different prime factors of the set are 2, 3, x, $(x - y)$, $(x + y)$, and $(x^2 + y^2)$. Hence the L.C.M. is $2^2 \cdot 3x^4(x - y)^2(x + y)^2(x^2 + y^2)$.

28. FRACTIONS

The same properties hold for fractions having algebraic expressions for numerators and denominators as were developed in Articles 15, 16, and 17, Chapter II.

Multiplication of Fractions

To multiply fractions, factor each numerator and each denominator except perhaps those expressions that are common to some numerator and some denominator. Then, employ cancellation until no numerator contains a factor common with any denominator. After cancellation multiply the remaining factors in the numerator for the numerator of the product and multiply the remaining factors in the denominator for the denominator of the product.

Cancellation *consists of dividing both numerator and denominator by the same factor.* To illustrate, dividing both numerator and denominator by $(x - y)$ in the following, we have

$$\frac{\overset{1}{\cancel{(x-y)}}}{\underset{1}{\cancel{(x-y)}}} = \frac{1}{1} = 1 \qquad \text{and} \qquad \frac{\overset{1}{\cancel{(x-y)}}}{\underset{(x-y)^2}{\cancel{(x-y)^3}}} = \frac{1}{(x-y)^2}. \tag{4}$$

In many cases the quotient 1 is not written but is understood.

Example 1. Find the product $\dfrac{x^2 - 3x + 9}{2x - 6} \times \dfrac{x^2 - 9.}{x^3 + 27}$.

Solution. Factoring each numerator and each denominator, and simplifying, we get

$$\frac{x^2 - 3x + 9}{2x - 6} \times \frac{x^2 - 9}{x^3 + 27}$$

$$= \frac{\cancel{x^2 - 3x + 9}}{2\cancel{(x-3)}} \times \frac{\cancel{(x-3)}(x+3)}{\cancel{(x+3)}\cancel{(x^2 - 3x + 9)}} = \frac{1}{2}.$$

Let us observe how to divide a product by a given number, and how to divide a sum or difference by a given number.

When a product is divided by a given number, only one member of the product is divided by the divisor, as:

$$4 \cdot 14 \div 2 = \frac{4 \cdot 14}{2} = 2 \cdot 14 \qquad \text{or} \qquad 4 \cdot 7,$$

and

$$ab \div c = \frac{ab}{c} = \frac{a}{c} \cdot b \qquad \text{or} \qquad a \cdot \frac{b}{c}. \tag{5}$$

When a sum or difference is divided by a given number, **each** member of the sum or each member of the difference must be divided by the divisor, as:

$$(4 + 14) \div 2 = \frac{\overset{2}{\cancel{4}} + \overset{7}{\cancel{14}}}{\cancel{2}} = 2 + 7 = 9$$

not

$$\frac{\overset{7}{\cancel{4 + 14}}}{\cancel{2}} \qquad \text{or} \qquad 11;$$

and

$$(a + b) \div c = \frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}. \tag{6}$$

However, to avoid any possible errors in cancellation, *it is best that the numerators and the denominators be written in factored form before cancellation is employed.* To illustrate,

$$\frac{4+14}{2} = \frac{2(2+7)}{2} = 9$$

is to be preferred to

$$\frac{\cancel{4}+\cancel{14}}{\cancel{2}} = 2 + 7 = 9.$$
$$\begin{matrix} 2 & 7 \end{matrix}$$

Division of Fractions

To divide fractions, factor all numerators and all denominators; invert all fractions in the divisor, and then, after cancellation, multiply the result.

Example 2. Simplify

$$\frac{x^2-y^2}{x^2-3xy+2y^2} \div \frac{x^2+xy}{xy-2y^2} \times \frac{x^2-xy}{(x-y)^2}.$$

Solution. Factoring each numerator and each denominator, inverting the divisor, and simplifying, we obtain

$$\frac{x^2-y^2}{x^2-3xy+2y^2} \div \frac{x^2+xy}{xy-2y^2} \times \frac{x^2-xy}{(x-y)^2}$$
$$= \frac{\cancel{(x-y)}\cancel{(x+y)}}{\cancel{(x-2y)}(x-y)} \times \frac{y\cancel{(x-2y)}}{x\cancel{(x+y)}} \times \frac{x\cancel{(x-y)}}{\cancel{(x-y)^2}} = \frac{y}{x-y}.$$

Addition and Subtraction of Fractions

To add or subtract fractions, the following seven steps should be followed carefully.

STEP 1. *Rewrite the given problem so that all the numerators and all the denominators are arranged in ascending or descending powers of some common letter, and write each fraction with its denominator factored.*

STEP 2. *Express each fraction with the same common denominator. This is accomplished by multiplying the numerator of each fraction by the quotient obtained by dividing the least common denominator (L.C.D.) by the denominator of this fraction. It is best to indicate all such products.*

STEP 3. *Add the new numerators. This addition should be indicated.*

STEP 4. *Enclose the separate products in parentheses unless the products are very simple. In case the products are simple, Step 4 may be omitted.*

STEP 5. *Remove all parentheses.*

STEP 6. *Collect like terms.*

STEP 7. *Check result by assigning values to the different letters appearing in the given fractions and in their sum.*

When adding fractions, never fail to write the common denominator of the sum, and be certain that this denominator is the *least common denominator* and not just a common denominator.

Example 3. Add

$$\frac{x^2+2}{x^2-1} - \frac{x+2}{1+x} + \frac{3-2x}{(1-x)^2} - 2.$$

Solution. Arranging each numerator and each denominator in decending powers of x, factoring each denominator, and observing that the least common denominator is $(x+1)(x-1)^2$, we have

$$\frac{x^2+2}{x^2-1} - \frac{x+2}{1+x} + \frac{3-2x}{(1-x)^2} - 2$$

$$= \frac{x^2+2}{(x-1)(x+1)} - \frac{x+2}{x+1} - \frac{2x-3}{(x-1)^2} - 2 \qquad \text{(Step 1)}$$

$$= \frac{(x^2+2)(x-1)}{(x+1)(x-1)^2} - \frac{(x+2)(x-1)^2}{(x+1)(x-1)^2} - \frac{(2x-3)(x+1)}{(x+1)(x-1)^2} - \frac{2(x+1)(x-1)^2}{(x+1)(x-1)^2} \qquad \text{(Step 2)}$$

$$= \frac{\left[\begin{array}{r}(x^2+2)(x-1) - (x+2)(x-1)^2 - \\ (2x-3)(x+1) - 2(x+1)(x-1)^2\end{array}\right]}{(x+1)(x-1)^2} \qquad \text{(Step 3)}$$

$$= \frac{\left[\begin{array}{r}(x^3-x^2+2x-2) - (x^3-3x+2) - \\ (2x^2-x-3) - 2(x^3-x^2-x+1)\end{array}\right]}{(x+1)(x-1)^2} \qquad \text{(Step 4)}$$

$$= \frac{\left(\begin{array}{r}x^3-x^2+2x-2-x^3+3x-2- \\ 2x^2+x+3-2x^3+2x^2+2x-2\end{array}\right)}{(x+1)(x-1)^2} \qquad \text{(Step 5)}$$

$$= \frac{-2x^3-x^2+8x-3}{(x+1)(x-1)^2}. \qquad \text{(Step 6)}$$

The result obtained by adding or subtracting algebraic fractions may be checked by assigning a value to each of the different letters involved and then substituting these values for the letters in each of the original fractions and in the obtained result. If the sum

and difference of the numerical fractions is the same as the value of the obtained result, the solution is probably correct. Any values can be assigned to the letters except 0 and 1, and those values which will make any one of the given denominators zero. Thus, Example 3 can be checked by letting x equal any number except 1 or -1.

Let $x = 2$; then

$$\frac{6}{3} - \frac{4}{3} + \frac{-1}{1} - 2 = \frac{-16 - 4 + 16 - 3}{3 \cdot 1}; \qquad \text{(Step 7)}$$

adding, we obtain

$$-\tfrac{7}{3} = -\tfrac{7}{3},$$

which indicates that the sum is correct.

29. COMPLEX FRACTIONS

A fraction whose numerator or whose denominator, or both numerator and denominator, contain fractions is called a complex fraction, as:

$$\frac{1 + \dfrac{1}{x-1}}{1 - \dfrac{1}{x+1}}.$$

We shall give two methods for simplifying complex fractions.

FIRST METHOD. Express both numerator and denominator of the given fraction as ordinary fractions, and then simplify as the quotient of two fractions. To illustrate:

Example. Simplify $\dfrac{2 - \dfrac{2x-5}{x-1}}{\dfrac{5x^2-3}{x^2-1} - 4}$.

Solution. Adding the numerator and adding the denominator gives

$$\frac{2 - \dfrac{2x-5}{x-1}}{\dfrac{5x^2-3}{x^2-1} - 4} = \frac{\dfrac{2(x-1) - (2x-5)}{x-1}}{\dfrac{(5x^2-3) - 4(x^2-1)}{x^2-1}} = \frac{\dfrac{3}{x-1}}{\dfrac{x^2+1}{x^2-1}}$$

$$= \frac{3}{x-1} \cdot \frac{(x-1)(x+1)}{x^2+1} = \frac{3(x+1)}{x^2+1}.$$

SECOND METHOD. Multiply both numerator and denominator of the given fraction by the L.C.D. of all the denominators in both

numerator and denominator. To illustrate, in the same example multiply both numerator and denominator by $x^2 - 1$.

Example. Simplify $\dfrac{2 - \dfrac{2x - 5}{x - 1}}{\dfrac{5x^2 - 3}{x^2 - 1} - 4}$.

Solution.

$$\frac{2 - \dfrac{2x - 5}{x - 1}}{\dfrac{5x^2 - 3}{x^2 - 1} - 4} = \frac{2(x^2 - 1) - (2x - 5)(x + 1)}{(5x^2 - 3) - 4(x^2 - 1)}$$

$$= \frac{2x^2 - 2 - (2x^2 - 3x - 5)}{5x^2 - 3 - 4x^2 + 4} = \frac{3x + 3}{x^2 + 1}.$$

QUESTIONS

1. What is meant by the least common multiple?
2. How are fractions multiplied?
3. How are fractions divided?
4. How are fractions added?

EXERCISES

Simplify the following fifteen expressions:

1. $\dfrac{3}{x - 3} - \dfrac{1}{x + 2}$.

2. $\dfrac{3}{x - 3} - \dfrac{1}{x + 2} + 1$.

3. $\dfrac{a}{a^2 - 1} - \dfrac{1}{a - 1}$.

4. $\dfrac{10ax^2}{7b^3y} \times \dfrac{21by^3}{45a^2x}$.

5. $\dfrac{x^2 - 25}{x^2 + 3x} \times \dfrac{x^2 - 9}{x^2 - 5x}$.

6. $\dfrac{27x^3a}{15b^2y^3} \div \dfrac{18x^2a^2}{20b^2y^2}$.

7. $\dfrac{a^2 + b^2}{a^2 - b^2} \div \dfrac{a + b}{a - b}$.

8. $\dfrac{6x^2 - 4x - 10}{2}$.

9. $\dfrac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$.

10. $\dfrac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$.

11. $\dfrac{x^2 + xy - 6y^2}{6x^2 - xy - y^2} \div \dfrac{x - 2y}{2x - y}$.

12. $\dfrac{x^2 + 3x + 9}{3x + 9} \times \dfrac{x^2 - 9}{x^3 - 27}$.

13. $\dfrac{(a^2 - b^2) + x(a - b)}{a - b}$.

14. $\dfrac{2x^4 - x^2 - 1}{x^2 - 1}$.

15. $\dfrac{36 + 16}{4 + 8}$.

Find the H.C.F. and the L.C.M. of each of the expressions in Exercises 16 through 24.

16. 2, 4, 6, 8, 10.

17. 12, 40, 80, 100, 120.

18. a^2xy^3, ax^2y, bx^3y^2, b^4xy^4.

19. $2a^2x^2$, $6a^3x^3$, $12ax^2y$, $18a^2x^3y^2$.

20. $(x-y)$, (x^2-y^2), (x^3-y^3).

21. (x^2-y^2), $(x-y)^2$, $(x+y)^2$.

22. $(x-y)(y-z)$, $(y-z)(z-x)$, $(z-x)(x-y)$.

23. $21x(xy-y^2)^2$, $35(x^4y^2-x^2y^4)$, $15y(x^2+xy)^2$.

24. $x^2+y^2-z^2+2xy$, $x^2-y^2+z^2+2xz$, $x^2-y^2-z^2-2yz$.

In Exercises 25 through 84, perform the indicated operations and simplify the results. The results may be checked by assigning values to the letters.

25. $\dfrac{3x-2y}{4}-\dfrac{y+2x}{6}-\dfrac{3y-2x}{8}$.

26. $\dfrac{2x}{x^2-4}-\dfrac{x}{2-x}+\dfrac{5}{2+x}$.

27. $2x-\dfrac{3}{x-1}+\dfrac{1}{x}-\dfrac{2x}{1-x}$.

28. $\dfrac{x+1}{x^2-1}-\dfrac{1-2x}{(1-x)^2}$.

29. $\dfrac{x}{3}-2x+\dfrac{5}{x-1}-\dfrac{3x}{6(1-x)}$.

30. $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{y-2x}{y^2-x^2}$.

31. $\dfrac{x^2}{x^3+y^3}-\dfrac{y^2}{x^3-y^3}+\dfrac{x^3y^2+x^2y^3}{x^6-y^6}$.

32. $\dfrac{2x+y}{2x-y}-\dfrac{3x-y}{3x+y}-\dfrac{6x^2-xy+y^2}{6x^2-xy-y^2}$.

33. $\dfrac{x+3}{x-2}-\dfrac{x+2}{x-3}+\dfrac{1}{x+1}$.

34. $\dfrac{2-3x}{3x+2}+\dfrac{x-2}{3-2x}-\dfrac{3-4x^2+5}{6x^2-5x-6}$.

35. $\dfrac{x+2y}{x-2y}-\dfrac{x-2y}{x+2y}-\dfrac{8y}{(x-2y)^2}$.

36. $\dfrac{x^2-3x}{x^2-4}-\dfrac{1-x}{2-x}+\dfrac{2x-1}{x+2}-\dfrac{x^2-2x-3}{(2-x)^2}$.

37. $\dfrac{3x}{x+a}-\dfrac{6a}{x^2-a^2}+\dfrac{3x}{a-x}-\dfrac{6a}{x^2+a^2}$.

38. $\dfrac{x}{3+x}-2x-\dfrac{x}{x-3}+1$.

39. $\dfrac{b}{(a-b)(b-c)}+\dfrac{c}{(b-c)(c-a)}+\dfrac{a}{(c-a)(a-b)}$.

40. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-a)(b-c)}+\dfrac{c}{(c-a)(c-b)}$.

41. $\dfrac{2}{2x^2-3x}-\dfrac{3}{2x^2}-\dfrac{2x-3}{-3x}$.

42. $\dfrac{2a-3b}{2a+3b}+\dfrac{2b+3a}{2b-3a}-\dfrac{6a^2-5ab+6b^2}{6a^2+5ab-6b^2}$.

43. $\dfrac{2}{x-2}-\left(\dfrac{3x}{4-x^2}-\dfrac{2-3x}{x+2}\right)$.

44. $\dfrac{a-b}{(a-b)^2-c^2}-\dfrac{a+b}{(a-c)^2-b^2}$.

45. $\dfrac{10a^2+30ab}{25a^2-9b^2}+\dfrac{ab+21b^2}{9b^2-25a^2}$.

46. $\dfrac{9a^2-b^2}{c^2-9a^2}+\dfrac{b+c}{c+3a}-\dfrac{b-c}{3a-c}$.

47. $\dfrac{a^2-b^2}{a^2+b^2}\div\dfrac{(a-b)^2}{a+b}\times\dfrac{a^2-b^2}{(a+b)^2}$.

48. $\dfrac{x^3-8y^3}{x^2-xy}\div\dfrac{x^2+2xy+4y^2}{x^2-xy+y^2}\times\dfrac{x^3-xy^2}{x^3+y^3}$.

49. $\dfrac{x^3-2x^2y+xy^2}{x^2y-y^3}\times\dfrac{x^3-y^3}{x^2-xy}\div\dfrac{x^2+xy+y^2}{xy+y^2}$.

50. $\dfrac{(x+y)^2-z^2}{x^2+xy-xz}\times\dfrac{x^2y^2z^2}{x^2+xy+xz}\div\dfrac{y^2z^2}{xyz}$.

51. $\dfrac{2x^2-x-6}{x^4+x^2+1}\div\dfrac{2x^2+x-3}{x^3-1}\times\dfrac{x^3+1}{x^2-3x+2}$.

52. $\dfrac{a^3-b^3}{a^3}\times\dfrac{b^2-a^2}{(a-b)^2}\div\dfrac{a^2b+ab^2+b^3}{a^2+2ab+b^2}$.

53. $\dfrac{a^4-b^4}{(a+b)^2}\times\dfrac{a^2-b^2}{(a-b)^2}\times\dfrac{a+b}{a^2+b^2}$.

54. $\left(\dfrac{a+b}{a-b}-\dfrac{a^3+b^3}{a^3-b^3}\right)\div\left(\dfrac{a+b}{a-b}-\dfrac{a^2+b^2}{a^2-b^2}\right)$.

55. $\left(y+\dfrac{xy}{y-x}\right)\left(x-\dfrac{xy}{x+y}\right)$.

56. $\left(x+\dfrac{xy}{x-y}\right)\left(y-\dfrac{xy}{x+y}\right)$.

57. $\left(1-\dfrac{x^2}{y^2}\right)\left(\dfrac{x}{x-y}-1\right)$.

58. $\dfrac{x^3-y^3}{x+2y}\cdot\dfrac{x^2+3xy+2y^2}{x^3+y^3}$.

59. $\left(\dfrac{x}{y}-\dfrac{x-2y}{x-y}\right)\div\left(\dfrac{x}{y}-2-\dfrac{x}{y-x}\right)$.

60. $\left(\dfrac{x}{1-\dfrac{1}{x}}-x+\dfrac{1}{1-x}\right)\div\left(\dfrac{x}{1+\dfrac{1}{x}}+x-\dfrac{1}{1+x}\right)$.

61. $\left(\frac{x}{2} - \frac{2}{x}\right) \div \left(4 - \frac{2x}{x+2}\right)$.

62. $\frac{(a-b)^2 - c^2}{a^2 - (b-c)^2} \times \frac{(a+b)^2 - c^2}{a^2 - (b+c)^2}$.

63. $\left(1 - \frac{3-2x}{1-x} - \frac{2x^2+6}{x^2-1}\right) \div \left(1 + \frac{1+2x}{1-x} + \frac{2+x^2}{1+x^2}\right)$.

64. $(2x-5)\,\frac{2x+5}{10+x-2x^2}$.

65. $(x^2 - 4y^2)\left(\frac{x-2y}{x+2y}\right)$.

66. $\left(1 - \frac{x^2}{y^2}\right)\left(\frac{x}{x-y} - 1\right) - \left(1 - \frac{x^3}{y^3}\right)\left(1 - \frac{xy+y^2}{x^2+xy+y^2}\right)$.

67. $\dfrac{x - \frac{1}{x}}{x + \frac{1}{x}}$.

68. $\dfrac{\frac{b}{a} + 1}{\frac{a^2}{b^2} - 1}$.

69. $\dfrac{\frac{x^2}{y} + \frac{y^2}{x}}{x + \frac{y^2}{x} - y}$.

70. $\dfrac{\frac{a+b}{c-d} - \frac{a-b}{c+d}}{\frac{a-b}{c+d} - \frac{a+b}{c-d}}$.

71. $\dfrac{2 - \frac{3}{x-2}}{5 - \frac{2x^2-5x+4}{2x^2-3x-2}}$.

72. $\dfrac{\frac{2}{x} - \frac{x-1}{x+2} + \frac{x+2}{x-1}}{\frac{x}{2} - \frac{x+2}{x-1} + \frac{x-1}{x+2}}$.

73. $\dfrac{\frac{1}{x} - x^2}{x + \frac{1}{x} + 1}$.

74. $\dfrac{\frac{a^2+b^2}{b} - a}{\frac{a^2+b^2}{b} - \frac{ab-b^2}{a}}$.

75. $\dfrac{1}{x + \frac{1}{x - \frac{1}{x}}}$.

76. $\dfrac{\frac{1}{\sqrt{x^3 - y^3}}}{1 - \frac{x}{x + \frac{y^2}{x+y}}}$.

77. $\dfrac{1}{x - \frac{1}{x + \frac{1}{x}}}$.

78. $\dfrac{1 - \frac{3x}{3x-2y}}{\frac{3 + \frac{2y}{x}}{3 - \frac{2x}{y}} - \frac{2 + \frac{3y}{x}}{2 - \frac{3x}{y}}}$.

79. $1 - \dfrac{2}{2 - \dfrac{3}{1 - \frac{1}{5}}}$.

80. $\dfrac{\frac{1}{3}}{\frac{1}{6} + \dfrac{\frac{1}{9}}{\frac{1}{12} + \dfrac{\frac{1}{15}}{\frac{1}{18} + \frac{1}{21}}}}$

81. $\dfrac{(a^2 - x^2)^{\frac{1}{2}} + x^2(a^2 - x^2)^{-\frac{1}{2}}}{(a^2 - x^2)^{\frac{1}{2}}}$.

82. $\dfrac{a(x - a)^{\frac{1}{2}} - x(x - a)^{-\frac{1}{2}}}{x - a}$.

83. $\dfrac{(x^2 - a^2)^{\frac{2}{3}} - x^2(x^2 - a^2)^{-\frac{1}{3}}}{(x^2 - a^2)^{\frac{2}{3}}}$.

84. $\dfrac{(a^2 - x^2)^{\frac{1}{2}} - x \cdot \frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x)}{a^2 - x^2}$.

CHAPTER V

Logarithms

THEOREMS OF LOGARITHMS

I. $\log_a M \cdot N = \log_a M + \log_a N.$

II. $\log_a \frac{M}{N} = \log_a M - \log_a N.$

III. $\log_a M^p = p \log_a M.$

IV. $\log_a \sqrt[n]{M} = \frac{1}{n} \log_a M.$

30. PROPERTIES OF LOGARITHMS

If the values of the different powers of a particular number are computed and put in table form, then by the first three laws in Article 21, Chapter III, the operations of multiplication, division, extraction of roots, and finding powers of numbers may be accomplished by addition or subtraction or by simple multiplication, provided the result is a number in the table. For example, suppose we have a table of values of the powers of 2 and wish to find the value of the product 128×512; then, from the table we find $128 = 2^7$ and $512 = 2^9$. Hence, $128 \times 512 = 2^7 \cdot 2^9 = 2^{16}$, and from the table we find $2^{16} = 65{,}536$, which is the value of the product 128×512. Hence, the computation necessary to find the value of the product 128×512 has been reduced to adding 7 and 9, and the product can be found in the table. These exponents are called logarithms, and the 2 is called the base. Thus, 7 is the logarithm of 128 to the base 2 and we write $\log_2 128 = 7$; also $\log_2 512 = 9$ and $\log_2 65{,}536 = 16$.

DEFINITION. *The logarithm of a positive number* N *to a given positive base* a $(a \neq 1)$ *is the exponent of the power to which the base must be raised to equal* N*; that is,*

$$a^{\log_a N} = N, \tag{1}$$

where the symbol $\log_a N$ is read "*the logarithm of* N *to the base* a." Also if

$$a^x = N, \tag{2}$$

then x *is the logarithm of* N *to the base* a *and we write* $\log_a N = x$. Thus,

$$a^x = a^{\log_a N} = N. \tag{3}$$

It is very important to be able to express a logarithmic equation as an exponential equation, and vice versa. That is, it is important to see that

$$\log_a N = x \tag{4}$$

and

$$a^x = N \tag{5}$$

are two ways of expressing the same thing. For example,

$$\log_5 125 = 3 \tag{6}$$

also means

$$5^3 = 125. \tag{7}$$

Since logarithms are exponents, then by applying the laws of exponents in Article 21, we have the following properties:

(1) *The logarithm of the base is* 1.

This is true because $a^1 = a$, and hence $\log_a a = 1$.

(2) *The logarithm of* 1 *to any base is* 0.

This is true because $a^0 = 1$, hence $\log_a 1 = 0$.

(3) *Zero has no logarithm.*

This is true because no value of x can be found which will satisfy the equation $a^x = 0$.

(4) *For a positive base, a negative number has no real logarithm.*

This is true because if a is positive, a^x is a positive number for all real values of x.

THEOREM I. *The* ***logarithm of a product*** *of several positive numbers is equal to the sum of the logarithms of each of the numbers, all logarithms being taken to the same base.*

To prove this, let

$$M = a^x \quad \text{and} \quad N = a^y; \tag{8}$$

then by Equations (4) and (5),

$$\log_a M = x \quad \text{and} \quad \log_a N = y. \tag{9}$$

From Equation (8),

$$M \cdot N = a^x \cdot a^y = a^{x+y}; \tag{10}$$

then by Equations (4) and (5),

$$\log_a M \cdot N = x + y,$$

and by Equation (9),

$$x + y = \log_a M + \log_a N. \tag{11}$$

Therefore,

$$\log_a M \cdot N = \log_a M + \log_a N. \tag{12}$$

Illustration:

$$\log (268)(3.14) = \log 268 + \log 3.14.$$

THEOREM II. *The* ***logarithm of a quotient*** *of two positive numbers, if the quotient is written in fraction form, is equal to the logarithm of the numerator minus the logarithm of the denominator, all logarithms being taken to the same base.*

Proof: From Equation (8),

$$\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y}; \tag{13}$$

by (4), (5), and (9),

$$\log_a \frac{M}{N} = x - y = \log_a M - \log_a N. \tag{14}$$

Therefore,

$$\log_a \frac{M}{N} = \log_a M - \log_a N. \tag{15}$$

Illustration:

$$\log \frac{375}{26} = \log 375 - \log 26.$$

THEOREM III. *The* ***logarithm of the pth power*** *of a positive number is equal to p times the logarithm of the number, all logarithms being taken to the same base.*

Proof: From Equation (8),

$$M^p = (a^x)^p = a^{px}; \tag{16}$$

by (4), (5), and (9),

$$\log_a M^p = px = p \log_a M. \tag{17}$$

Therefore,

$$\log_a M^p = p \log_a M. \tag{18}$$

Illustration:

$$\log (0.265)^4 = 4 \log 0.265.$$

For the extraction of roots, we may write Theorem III as follows:
THEOREM IV. *The logarithm of any root of a positive number is found by dividing the logarithm of the number by the index of the root.* Therefore,

$$\log_a \sqrt[n]{M} = \log_a M^{1/n} = \frac{1}{n} \log_a M. \tag{19}$$

Illustration:

$$\log \sqrt[3]{168^4} = \frac{4}{3} \log 168.$$

We shall illustrate the applications of the four preceding theorems to express the logarithm of a given expression in terms of the logarithms of the separate numbers that appear in the given expression.

Example 1. Express $\log \sqrt[5]{\dfrac{11^2 \cdot \sqrt[3]{13} \cdot 7^{3/4}}{19^3 \cdot \sqrt[5]{17^4} \cdot 5^7}}$ in terms of the logarithms of separate numbers.

Solution.

$$\log \sqrt[5]{\frac{11^2 \cdot \sqrt[3]{13} \cdot 7^{3/4}}{19^3 \cdot \sqrt[5]{17^4} \cdot 5^7}}$$

$$= \frac{1}{5} \log \left[\frac{11^2 \cdot \sqrt[3]{13} \cdot 7^{3/4}}{19^3 \cdot \sqrt[5]{17^4} \cdot 5^7} \right]. \qquad \text{(Theorem IV)}$$

$$= \frac{1}{5} [\log (11^2 \cdot \sqrt[3]{13} \cdot 7^{3/4}) - \log (19^3 \cdot \sqrt[5]{17^4} \cdot 5^7)] \qquad \text{(Theorem II)}$$

$$= \frac{1}{5} [(\log 11^2 + \log \sqrt[3]{13} + \log 7^{3/4}) -$$

$$(\log 19^3 + \log \sqrt[5]{17^4} + \log 5^7)] \qquad \text{(Theorem I)}$$

$$= \frac{1}{5} \left[\left(2 \log 11 + \frac{1}{3} \log 13 + \frac{3}{4} \log 7 \right) - \right.$$

$$\left. \left(3 \log 19 + \frac{4}{5} \log 17 + 7 \log 5 \right). \right. \qquad \text{(Theorem III)}$$

However, one should be able to apply these theorems directly, and thus be able to write immediately

$$\log \sqrt[5]{\frac{11^2 \cdot \sqrt[3]{13} \cdot 7^{3/4}}{19^3 \cdot \sqrt[5]{17^4} \cdot 5^7}}$$

$$= \frac{1}{5}\left[\left(2 \log 11 + \frac{1}{3} \log 13 + \frac{3}{4} \log 7\right) - \left(3 \log 19 + \frac{4}{5} \log 17 + 7 \log 5\right)\right].$$

If the logarithm of a given quantity is expressed as a sum and difference of logarithms of other quantities, it is possible to write the functional relation connecting these quantities. To illustrate:

Example 2. Given $\log T = \log 2 + \log \pi + \frac{1}{2} \log l - \frac{1}{2} \log g$. Find T.

Solution. It is apparent that the formula for T is

$$T = 2\pi \sqrt{\frac{l}{g}}.$$

To solve for an unknown in a logarithmic equation, it is helpful to write the equation as an exponential equation and then solve for the unknown. To illustrate:

Example 3. Find the value of $\log_3 81$.

Solution. Let

$$\log_3 81 = x;$$

then the exponential form is

$$3^x = 81 = 3^4;$$

hence,

$$x = 4.$$

Example 4. Given $\log_a 32 = -5$. Find a.

Solution. If

$$\log_a 32 = -5,$$

then the exponential form is

$$a^{-5} = 32 = 2^5 = (\tfrac{1}{2})^{-5};$$

hence,

$$a = \tfrac{1}{2}.$$

QUESTIONS

1. Define logarithm.
2. Give four properties of logarithm.
3. Give four theorems for operating with logarithms and prove each theorem.

EXERCISES

Find the value of each of the following twelve expressions:

1. $\log_2 8$. **2.** $\log_8 2$. **3.** $\log_{10} 100$. **4.** $\log_{10} 0.0001$.

5. $\log_2 \frac{1}{8}$. **6.** $\log_8 \frac{1}{2}$. **7.** $\log_{10} 1$. **8.** $\log_{10} 1{,}000$.

9. $\log_4 8$. **10.** $\log_8 32$. **11.** $\log_{10} 0.01$. **12.** $\log_{10} 10$.

Find the value of the base in each of the following six expressions:

13. $\log_a 81 = 4$. **14.** $\log_a 100 = -2$. **15.** $\log_a 125 = -3$.

16. $\log_a 4 = \frac{1}{3}$. **17.** $\log_a 125 = 3$. **18.** $\log_a \frac{1}{32} = 5$.

Give the value of the letter in each of the following exercises (19 through 57):

19. $\log_9 3 = x$. **20.** $\log_a 9 = 3$. **21.** $\log_3 N = 4$.

22. $\log_{\frac{1}{2}} 8 = x$. **23.** $\log_a 3 = 9$. **24.** $\log_4 \frac{1}{2} = x$.

25. $\log_a 125 = -\frac{3}{2}$. **26.** $\log_2 N = -5$. **27.** $\log_3 9 = x$.

28. $\log_a 10{,}000 = 2$. **29.** $\log_5 125 = x$. **30.** $\log_9 N = \frac{1}{2}$.

31. $\log_{125} 5 = x$. **32.** $\log_a 0.0001 = -2$. **33.** $\log_{10} N = 4$.

34. $\log_a 16 = \frac{4}{5}$. **35.** $\log_{16} 32 = x$. **36.** $\log_{10} N = -3$.

37. $\log_{32} 16 = x$. **38.** $\log_a 243 = \frac{5}{4}$. **39.** $\log_{10} 0.001 = x$.

40. $\log_{10} N = 5$. **41.** $\log_a \frac{4}{9} = \frac{2}{3}$. **42.** $\log_{10} 10{,}000 = x$.

43. $\log_a \frac{4}{9} = -\frac{2}{3}$. **44.** $\log_{16} N = \frac{3}{4}$. **45.** $\log_{10} 0.01 = x$.

46. $\log_a 0.008 = 3$. **47.** $\log_{10} 0.1 = x$. **48.** $\log_{\frac{1}{2}} N = -5$.

49. $\log_a 36 = 2$. **50.** $\log_{10} \dfrac{1}{1{,}000} = x$. **51.** $\log_8 N = \frac{2}{3}$.

52. $\log_2 \frac{1}{64} = x$. **53.** $\log_a 2 = 1$.

54. $\log_{0.1} 0.0001 = x$. **55.** $\log_6 N = -2$.

56. $\log_{0.1} 1{,}000 = x$. **57.** $\log_a 0.0001 = -4$.

Express the logarithm of each of the following expressions (Exercises 58 through 84) in terms of the logarithms of the separate numbers.

58. $\log (11)^{\frac{3}{2}}(21)^{\frac{2}{3}}$. **59.** $\log \frac{4}{3}\pi r^3$. **60.** $\log 7^3 \sqrt[5]{13^2}$.

61. $\log \dfrac{5\sqrt{35}}{12^2}$. **62.** $\log \dfrac{71(15)^2}{4^3}$. **63.** $\log \left[\dfrac{7^8 \sqrt[7]{19^5(13\pi)^{\frac{2}{3}}}}{23^3 \cdot 17^4}\right]^9$.

64. $\log \sqrt[5]{7 \cdot 12}$. **65.** $\log (5\pi)^2$. **66.** $\log \dfrac{1}{23^5}$.

67. $\log (35 \div 22)$. **68.** $\log 31^5 \sqrt[7]{67}$. **69.** $\log 37^6 \dfrac{\sqrt[3]{23^4 47^5}}{17^8}$.

70. $\log \dfrac{8 \cdot 13}{25}$. **71.** $\log 2\pi \sqrt{\dfrac{l}{g}}$. **72.** $\log \dfrac{\sqrt[3]{65}}{\sqrt{29}}$.

73. $\log \dfrac{15\sqrt[3]{71}}{(201)(67)^2}$.

74. $\log \dfrac{17\sqrt[5]{52}}{(112)^7}$.

75. $\log \dfrac{(11)^3\sqrt{78}}{(7)(5^6)\sqrt[4]{22}}$.

76. $\log \left[\dfrac{5\sqrt{21}}{6\sqrt[3]{15}}\right]^5$.

77. $\log \dfrac{8 \cdot 11 \cdot 107 \cdot 205}{65 \cdot 72}$.

78. $\log \dfrac{(35)^2(102)^{2/3}}{(65)^5(98)^{3/4}}$.

79. $\log \dfrac{\sqrt[3]{(24)^2\left(\frac{4}{7}\right)^5}}{\left[(21)^3\sqrt{\frac{7}{2}\frac{1}{2}}\right]^5}$.

80. $\log \dfrac{(23)^4\sqrt{(76)^5}}{[(85)^2(91)^{-3/4}]^5}$.

81. $\log \dfrac{(27)^4\sqrt[7]{(28)^2(101)^{-3}}}{(61)^5(84)^{-4}}$.

82. $\log \left[\dfrac{\sqrt{275(678)^{-3}}}{[(98)^3\sqrt[3]{67}]^5} \div \dfrac{(0.068)(0.02)^7}{(21)^2\sqrt[3]{37}}\right]$.

83. $\log \dfrac{(\sqrt[9]{367 \div 231})(67 \div 23)^7}{\sqrt[3]{(123)^5} \div \sqrt[8]{(431)^3}}$.

84. $\log \dfrac{\dfrac{(28)^3}{\sqrt{66}}}{\left(\frac{6}{2}\frac{5}{1}\right)^4} \div \dfrac{\sqrt[3]{\frac{2}{3}\frac{6}{5}}}{\left(\frac{8}{9}\frac{1}{2}\right)^{-3}}$.

In the next four exercises express x in terms of the given numbers and letters.

85. $\log x = \log 2 - 3 \log 5 + \log 7$.

86. $\log x = 2 \log 37 + \frac{1}{5}(2 \log 7 + \frac{3}{4} \log 11 - \frac{2}{3} \log 17)$.

87. $\log x = 2 \log a - \frac{3}{4} \log b - \frac{1}{3} \log c + 5 \log d$.

88. $\log x = \frac{2}{3}(\log a - 5 \log b) - \frac{3}{5}(2 \log c - 3 \log d)$.

89. The area A of a circle is $A = \pi r^2$. Find $\log A$ in terms of the logarithms of π and r.

90. The time T of a complete swing of a pendulum of length l is $T = 2\pi\sqrt{\dfrac{l}{g}}$. (a) Find $\log T$ in terms of the logarithms of the other letters. (b) Find $\log l$ in terms of the logarithms of the other letters.

91. The velocity V with which gas effuses through an opening is $V = \dfrac{2p}{d}$, where p is the difference of pressure on the two sides of the opening and d is the density of the gas. Find $\log V$ in terms of the logarithms of p and d.

92. The stretch S of a wire of length l and radius r by a weight m is $S = \dfrac{mgl}{\pi r^2 k}$, where k is a constant for a given material and is called

Young's modulus. (a) Find log S in terms of the logarithms of the other letters. (b) Find log r in terms of the logarithms of the other letters.

93. Poiseuille's law states that the quantity Q of liquid passing through a narrow bore is $Q = \frac{\pi p r^4}{8\,ln} T$, where the difference of pressure between the two ends is p, the length of the tube is l, the radius of the bore is r, the time is T, and the coefficient of viscosity is n. (a) Find log Q in terms of the logarithms of the other letters. (b) Find log T in terms of the logarithms of the other letters.

31. COMMON LOGARITHMS

Any positive real number except 0 and 1 can be used as the base of a system of logarithms. Zero and 1 cannot be used as the base of logarithms because any power of zero is zero and any power of 1 is 1. In numerical calculations, because of our decimal system, it is more convenient to have all numbers represented as powers of 10. Such logarithms are called **common logarithms.** Another very important system of logarithms is that of **natural logarithms,** which have as a base the number e, where $2.7182818 \cdots$ is an approximation to the transcendental number e and we write

$$e = 2.7182818 \cdots .$$

The natural logarithms are used extensively in scientific discussions.

For convenience when writing logarithms to the base 10, we shall omit writing the base. Thus, *we shall write* log 36 *instead of* $\log_{10} 36$.

In Article 6, our attention was called to the fact that the value of x which satisfies the equation $10^x = 2$ is an irrational number. This value of x is the logarithm of 2. The logarithms of most numbers are irrational numbers. Thus, only an approximation to the logarithms can be found when they are expressed as decimals, but this approximation can be found accurately to as many decimal places as we please. (See Article 36.) For the average computation a logarithm table of five or six decimal places is sufficient. *Computation of products and quotients by logarithms will not give results correct to any more digits than the number of digits in the decimal part of the logarithms used.*

In this course we shall explain the use of a four-place table of logarithms. It will not be difficult to adapt these methods to the use of a five-place or a six-place table of logarithms.

Every logarithm consists of two parts, an integral part called the **characteristic,** and a *positive* decimal part called the **mantissa.** The mantissas are given in any table of common logarithms and they are the logarithms of numbers between 1 and 10. The characteristic is the number that must be added to, or subtracted from, the mantissa to give the logarithm of a number greater than 10 or the logarithm of a positive number less than 1, and it must be supplied by the student. The following discussion will illustrate some of the properties of the characteristic. Let us assume that log 2.36 can be found in a table of logarithms, and thus assume for the time being that

$$\log 2.36 = 0.3729;$$

then

$$2.36 = 10^{0.3729}.$$

Since

$$2{,}360 = 2.36 \times 10^3 = 10^{0.3729} \times 10^3 = 10^{0.3729+3}, \qquad (20)$$

it follows that

$$\log 2{,}360 = 0.3729 + 3 = 3.3729.$$

Thus, the logarithm of 2,360 is found by adding 3 to the logarithm of 2.36. The 3 is the characteristic and .3729 is the mantissa of the logarithm of 2,360.

Furthermore, since

$$\begin{aligned}
2{,}360 &= 2.36 \times 10^3 &&= 10^{0.3729+3},\\
236 &= 2.36 \times 10^2 &&= 10^{0.3729+2},\\
23.6 &= 2.36 \times 10^1 &&= 10^{0.3729+1},\\
\mathbf{2.36} &= \mathbf{2.36} &&= \mathbf{10^{0.3729}},\\
0.236 &= 2.36 \times 10^{-1} &&= 10^{0.3729-1},\\
0.0236 &= 2.36 \times 10^{-2} &&= 10^{0.3729-2},\\
0.00236 &= 2.36 \times 10^{-3} &&= 10^{0.3729-3},
\end{aligned} \qquad (21)$$

it follows that the logarithms of these numbers are

$$\begin{aligned}
\mathbf{\log 2.36} &= \mathbf{0.3729}, & \mathbf{\log 2.36} &= \mathbf{0.3729},\\
\log 23.6 &= 1.3729, & \log 0.236 &= 0.3729 - 1,\\
\log 236 &= 2.3729, & \log 0.0236 &= 0.3729 - 2,\\
\log 2{,}360 &= 3.3729, & \log 0.00236 &= 0.3729 - 3.
\end{aligned} \qquad (22)$$

It is customary to write a positive characteristic to the left of the mantissa, as log 2,360 = 3.3729, but it would be confusing to

write a negative characteristic to the left of the mantissa. For example, if log 0.00236 is written as -3.3729, one is likely to think of this as meaning $-(3.3729)$ instead of $-3 + 0.3729$. Thus, a negative characteristic should always be written to the right of the mantissa, as $\log 0.00236 = 0.3729 - 3$. However, since we can add to and subtract from the characteristic any number we please without changing its value, it is possible to write a negative characteristic in many different ways. To illustrate, the following are a few of the different ways log 0.00236 may be written.

$$\begin{aligned} \log\ 0.00236 &= 0.3729 - 3, \\ &= 1.3729 - 4, \\ &= 5.3729 - 8, \\ &= 7.3729 - 10. \end{aligned}$$

This last representation is used very frequently.

A number is said to be expressed in **scientific notation** if written as a number between 1 and 10 multiplied by an indicated power of 10.

Example. Write 2,360 in scientific notation.

Solution.

$$2{,}360 = 2.36 \times 10^3.$$

The numbers appearing in the center column of Equations (21) are the numbers in the first column written in scientific notation.

From the preceding discussion we observe that:

The mantissa of $\log N$ *depends only on the order of the digits in the number* N *and is independent of the position of the decimal point.*

The characteristic of $\log N$ *depends only on the position of the decimal point in the number* N *and is independent of the digits representing* N.

The characteristic of $\log N$ *is equal to the exponent of* 10 *when the number* N *is written in scientific notation.*

Two rules for finding characteristics may be stated as follows:

RULE I. *The characteristic of the logarithm of a number* N *greater than* 1 *is positive and is* 1 *less than the number of digits in* N *to the left of the decimal point.*

RULE II. *The characteristic of the logarithm of a positive number* N *less than* 1 *is negative and is* $-k$ *if the first significant digit in* N *is in the* k*th decimal place.*

32. THE USE OF THE TABLE

A table of logarithms gives only the mantissas of the logarithms of numbers. That is, a logarithm table gives the logarithms of numbers between 1 and 10, and from these logarithms we can obtain the logarithm of a given number by adding the proper characteristic.

Since there are an infinite number of numbers between 1 and 10, it is impossible to have a logarithm table for all numbers. The mantissas of the logarithms of numbers with no more than three significant figures* can be read directly from a four-place table of logarithms. A four-place table of logarithms is given on pages 408 and 409 (Table II).

How to Find the Mantissa

To find, from the table, the mantissa of the logarithm of a number with three significant figures, locate the first two significant figures of the given number in the column headed n, then the mantissa is found to the right of these two digits and in the column headed by the third digit. To illustrate:

Example 1. Find log 2,360.

Solution. In the table of logarithms, locate the number 23 in the column headed n. In the row with 23 and in the column headed 6 is found the mantissa .3729. Since the given number 2,360 has four digits to the left of the decimal point, then by Rule I, the characteristic is 3. Thus,

$$\log 2{,}360 = 3.3729.$$

If the given number has less than three significant figures, we can form a sequence of three figures by adding zeros to the right of the sequence of figures in the given number. For example, the mantissa of the logarithm of a number with the sequence 24 is the same as the mantissa of the logarithm of a number with the sequence 240.

To find the mantissa of the logarithm of a number with four or five digits by using a four-place table of logarithms, it is necessary to use a process called **interpolation.** To interpolate, we assume

* In this connection by significant figures or significant digits we mean all the digits from the first one that is not zero to the last one that is not zero.

the change between two successive mantissas to be uniform and hence proportional to the change in the corresponding numbers. We shall illustrate the process of interpolation by the following example.

Example 2. Find the logarithm of 23.65.

Solution. Since there are two digits to the left of the decimal point, the characteristic of the logarithm of 23.65 is 1. The mantissa of the logarithm of 23.65 is midway between the mantissa of the logarithm of 23.6 and the mantissa of the logarithm of 23.7, since 23.65 is a number midway between the two numbers 23.6 and 23.7. From the table, the mantissa of log 23.6 is .3729 and the mantissa of log 23.7 is .3747. The difference D of these two mantissas is 18.* To obtain the mantissa of log 23.65, it is necessary to add 9 ($0.5 \times 18 = 9.0$) to .3729, the mantissa of 23.6. Thus,

$$\log 23.65 = 1.3738.$$

The work showing this calculation is as follows:

$$\begin{array}{rl} \log 23.6 = & 1.3729 \\ \log 23.7 = & \underline{1.3747} \\ D = & \quad 18 \\ & \quad .5 \\ & \overline{9.0} \end{array} \qquad \begin{array}{rl} \log 23.6 = & 1.3729 \\ & \quad 9 \\ \log 23.65 = & \overline{1.3738} \end{array}$$

To interpolate when finding the logarithm of a number with more than three digits, add to the mantissa of the logarithm of the first three digits 0.1 (*one tenth*) *of the product of the fourth digit* (*or remaining digits*) *and the difference* D, *between two successive mantissas appearing in the table.*

To illustrate, since

$$\log 23.6 = 1.3729,$$

and the difference D between the mantissa .3729 and the next larger mantissa .3747 appearing in the table is 18, then,

$$\left.\begin{array}{ll} \log 23.65 = 1.3738 & (.5 \times 18 = 9.0). \\ \log 23.67 = 1.3742 & (.7 \times 18 = 12.6, \text{ or } 13). \\ \log 23.63 = 1.3734 & (.3 \times 18 = 5.4, \text{ or } 5). \\ \log 23.637 = 1.3736 & (.37 \times 18 = 6.66, \text{ or } 7). \end{array}\right\} \quad (24)$$

* We need only use the numerical value of the difference between the mantissas, if we remember that this difference belongs to the fourth decimal place; that is, we write 18 instead of .0018.

How to Find the Antilogarithm

The process of finding the number corresponding to a given logarithm is the inverse of the process of finding the logarithm of the number; the number thus found is called the **antilogarithm** of the given logarithm. The mantissa of the given logarithm determines the sequence of the digits in the number; this sequence can be found from the table of logarithms. The characteristic determines the position of the decimal point in this sequence. The characteristic does not appear in the table.

If the mantissa of the given logarithm can be found in the table, then the first two digits of the required number are found in the column headed n and in the same row with the mantissa, and the third digit in the number is found at the top of the column containing the mantissa. To illustrate:

Example 3. Given $\log N = 1.3729$, find N.

Solution.

$$N = 23.6.$$

The characteristic 1 indicates that N has two digits to the left of the decimal point (Rule I).

If the mantissa of the given logarithm cannot be found in the table, it is necessary to use **interpolation** to find the number. To illustrate:

Example 4. Given $\log N = 1.3738$, find N.

Solution. In this case the mantissa .3738 cannot be found directly in the table of logarithms. The given mantissa .3738 lies between the two mantissas .3729 and .3747 which are given in the table. Thus, the first three digits of N are 236, the sequence corresponding to the mantissa .3729. Since the difference between .3729 and the given mantissa .3738 is 9, and the difference between the two successive mantissas .3729 and .3747 given in the table is 18, it follows that the number N is $\frac{9}{18}$ of the way from 23.6 to 23.7. Therefore,

$$N = 23.6\tfrac{9}{18} = 23.65.$$

To interpolate for the fourth digit when finding the number N *corresponding to a given logarithm, the fourth significant digit of* N *is* 10 *times the quotient formed by dividing the difference between the given mantissa and the mantissa of the logarithm corresponding to the first three digits in* N *by the difference* D. To illustrate:

$$\left.\begin{array}{l} \text{If } \log N = 1.3729, \text{ then } N = 23.6. \\ \text{If } \log N = 1.3738, \text{ then } N = 23.65; \\ \qquad\qquad 10(\tfrac{9}{18}) = 5. \\ \text{If } \log N = 1.3742, \text{ then } N = 23.67; \\ \qquad\qquad 10(\tfrac{13}{18}) = 7.2, \text{ or } 7. \\ \text{If } \log N = 1.3734, \text{ then } N = 23.63; \\ \qquad\qquad 10(\tfrac{5}{18}) = 2.8, \text{ or } 3. \\ \text{If } \log N = 1.3736, \text{ then } N = 23.64; \\ \qquad\qquad 10(\tfrac{7}{18}) = 3.9, \text{ or } 4. \end{array}\right\} \quad (25)$$

The number N cannot be expressed with more significant figures than there are decimal places in the logarithm table being used.

33. COLOGARITHMS

The logarithm of the reciprocal of a number is defined to be the cologarithm of the number. Thus,

$$\operatorname{colog} N = \log \frac{1}{N} = \log 1 - \log N$$
$$= 0 - \log N = -\log N. \quad (26)$$

To avoid negative mantissas in the cologarithms, it is customary to subtract $\log N$ from $10.0000 - 10$ when finding $\operatorname{colog} N$. Thus,

$$\operatorname{colog} N = -\log N = (10 - \log N) - 10. \quad (27)$$

To illustrate, find colog 26.57. Using the log value in the table and subtracting, we have

$$\begin{array}{rr} & 10.0000 - 10 \\ \log 26.57 = & \underline{1.4244} \\ \operatorname{colog} 26.57 = & 8.5756 - 10 \end{array} \quad (28)$$

Observe that colog N consists of those digits which, when added to the corresponding digits of $\log N$, will give 9 except for the last significant digit to the right, and this last sum must be 10. To illustrate:

$$\left.\begin{array}{ll} \text{If} & \log N = 3.7686, \\ \text{then} & \operatorname{colog} N = 6.2314 - 10. \\ \text{If} & \log N = 0.7686 - 4, \text{ or } 6.7686 - 10, \\ \text{then} & \operatorname{colog} N = 9.2314 - 6, \text{ or } 3.2314. \end{array}\right\} \quad (29)$$

Cologarithms are very useful when finding the logarithm of a quotient involving several factors in the denominator. To illustrate, find the logarithm of $\dfrac{A \cdot B \cdot C}{D \cdot E}$.

$$\log \frac{A \cdot B \cdot C}{D \cdot E} = \log A + \log B + \log C - (\log D + \log E), \tag{30}$$

or

$$\log \frac{A \cdot B \cdot C}{D \cdot E} = \log A + \log B + \log C + \operatorname{colog} D + \operatorname{colog} E.$$

In case cologarithms are not used, it is necessary to find the sum of the logarithms of the factors in the numerator and from this sum to subtract the sum of the logarithms of the factors in the denominator. If the cologarithms of the factors in the denominator are used, then the logarithm of a quotient is obtained by addition.

QUESTIONS

1. Define common logarithm.
2. Define characteristic.
3. Define mantissa.
4. Give two rules for determining the characteristic.
5. What is meant by interpolation?
6. What is meant by antilogarithm?
7. Give the method for interpolating when finding logarithms.
8. Give the method for interpolating when finding antilogarithms.

EXERCISES

1. Write each of the following (a) through (n), as a whole number or as a decimal number:

(a) 3×10^{3}. (b) 2.65×10^{4}. (c) 6.72×10^{-5}.
(d) 26.5×10^{-3}. (e) 0.3×10^{5}. (f) 3×10^{-10}.
(g) 37.6×10^{-3}. (h) 0.035×10^{-4}. (i) 0.006×10^{4}.

(j) The wave length of sodium light is $5{,}893 \times 10^{-8}$ centimeter.
(k) Light travels in one second 2.999×10^{10} centimeters.
(l) The pressure per square centimeter due to the radiation from the sun is 4.55×10^{-4} dyne.
(m) The diameter of a hydrogen molecule is 5.8×10^{-8} centimeter.
(n) The mass of the earth is 6×10^{21} tons.

2. Write each of the following numbers, (a) through (p), in scientific notation:

(a) For example, $2{,}675 = 2.675 \times 10^3$.

(b) 26,880. (c) 0.025. (d) 0.000687.

(e) 90,000,000. (f) 0.0000009. (g) 0.0924.

(h) 37.68. (i) 2.878. (j) 325.6.

(k) One kilowatt hour equals 10,000,000,000 ergs.

(l) The volume of the earth is 260,000 million cubic miles.

(m) The attractive force between the earth and the sun is 3,600,000 millions of millions of tons.

(n) The number of seconds in a year.

(o) The distance light travels in one year (186,330 miles per second).

(p) The distance to the North Star is 44 light-years. (One light-year is the distance light travels in one year.)

3. Give the characteristic of the logarithm of each of the following numbers.

(a) 26,880. (b) 0.025. (c) 2,768.5.

(d) 0.000687. (e) 90,000,000. (f) 0.2.

(g) 7.657. (h) 0.0924. (i) 0.0000009.

(j) 37.68. (k) 2.878. (l) 0.37.

4. Give the characteristic of the logarithm of each of the following numbers.

(a) 374.2. (b) .037. (c) 20.08.

(d) .00035. (e) 3.714. (f) .0904.

(g) .00002. (h) 10,001. (i) 2,037.

(j) 30,000. (k) 2.003. (l) 9,680.

5. Find the common logarithm of each of the following numbers.

(a) 3,740. (b) 8.42. (c) 0.638.

(d) 0.0237. (e) 1.05. (f) 8,000.

(g) 300. (h) 106. (i) 0.06.

6. Find the common logarithm of each of the following numbers.

(a) 28,000. (b) 3.07. (c) 0.024.

(d) 0.007. (e) 1.04. (f) 5,000.

(g) 0.0001. (h) 105. (i) 0.05.

7. Find the common logarithms of each of the following numbers.

(a) 2.678. (b) 3.689. (c) 26.75.

(d) 8,974. (e) 0.7522. (f) 0.0065.

(g) 2,003. (h) .06879. (i) 28.75.

8. Find the common logarithms of each of the following numbers.

(a) 0.00975. (b) 657.9. (c) 0.7568.
(d) 28.67. (e) 3,721. (f) 4,268.
(g) 37,890. (h) .02004. (i) 69.78.

9. Find the numbers whose common logarithms are given below.

(a) 0.6117. (b) 1.8451. (c) 2.9001.
(d) 3.4622. (e) 0.2568. (f) $0.4500 - 1$.
(g) 3.6845. (h) $0.0028 - 3$. (i) 6.2635.

10. Find the numbers whose common logarithms are given below.

(a) $0.3458 - 4$. (b) $0.8729 - 3$. (c) 4.4014.
(d) 0.3728. (e) 3.7289. (f) 1.2345.
(g) 2.8496. (h) $7.7200 - 10$. (i) $8.8600 - 10$.

11. Find the cologarithm of the numbers given in Exercise 7.
12. Find the cologarithm of the numbers given in Exercise 8.

34. COMPUTATION WITH LOGARITHMS

The use of logarithms simplifies the work for multiplication, division, raising to powers, and extraction of roots, but logarithms cannot be used directly to simplify the work for addition and subtraction.

The calculations by logarithms should always be arranged in a neat form. For convenience of adding, it is best to write the logarithms in column form with the equality signs in a column. To reduce the chance of error, and for convenience of checking results, the logarithm of every number should appear as a part of the solution. To illustrate, we should write $\log (0.056)^3$ as

$$\log (0.056)^3 = 3(0.7482 - 2) = 2.2446 - 6, \qquad (31)$$

instead of just writing $\log (0.056)^3 = 0.2446 - 4$. It is also desirable to represent any complex expression whose value is to be determined by some letter, say x.

The three major difficulties encountered when carrying out calculations by logarithms are *negative mantissas*, *negative fractional characteristics*, and *negative decimal characteristics*. In the next three examples we shall discuss how to avoid each of these difficulties.

Avoid Negative Mantissas

When subtracting logarithms, always write the logarithm in the minuend so that its positive part is greater than the positive part of the logarithm in the subtrahend. To illustrate:

Example 1. Find the value of $x = \dfrac{28.65}{3{,}786}$.

Solution.

$$\log x = \log 28.65 - \log 3{,}786.$$

Using values from the table and subtracting, we have

$$\begin{aligned} \log 28.65 &= 1.4571 = 11.4571 - 10 \\ \log 3{,}786 &= 3.5782 = \quad 3.5782 \\ \log x &= \quad 7.8789 - 10 \end{aligned}$$

Therefore,

$$x = 0.007567.$$

This illustrates how to subtract the logarithm of any number from the logarithm of a smaller number. Elsewhere in algebra, when we are asked to subtract a large number from a small number, we subtract the small number from the large number and attach the minus sign to the difference, as $3 - 5 = -2$. However, in logarithms negative mantissas should be avoided in the results, because only positive mantissas can be found in any logarithm table.

Avoid Negative Fractional Characteristics

When dividing a logarithm with a negative characteristic by a number p, write the characteristic so that the negative part of the characteristic is a multiple of p. To illustrate:

Example 2. Find the value of $\sqrt[3]{0.06872}$.

Solution. By Theorem IV

$$\log \sqrt[3]{0.06872} = \tfrac{1}{3} \log 0.06872 = \tfrac{1}{3}(0.8371 - 2).$$

However, the negative part of the characteristic in this case should be expressed as a multiple of 3. Thus, we write

$$\log \sqrt[3]{0.06872} = \tfrac{1}{3}(1.8371 - 3) \quad \text{or} \quad \tfrac{1}{3}(28.8371 - 30),$$

hence,

$$\log \sqrt[3]{0.06872} = 0.6124 - 1 \quad \text{or} \quad 9.6124 - 10.$$

Therefore,

$$\sqrt[3]{0.06872} = 0.4096.$$

Avoid Negative Decimal Characteristics

When finding the logarithm of a decimal fraction raised to a decimal power, write the negative part of the characteristic large enough so that in the product, the negative part of the characteristic will be a whole number. To illustrate:

Example 3. Find the value of $(0.028)^{.18}$.

Solution. By Theorem III

$$\begin{aligned}\log (0.028)^{.18} &= .18(98.4472 - 100)\\ &= 17.7205 - 18\\ &= 0.7205 - 1.\end{aligned}$$

Therefore,

$$(0.028)^{.18} = 0.5254.$$

However,

$$\log (0.028)^{.18} = .18(0.4472 - 2) = 0.0805 - .36$$

can be handled in the following manner. Adding and subtracting 1, we have

$$(1.0805 - .36) - 1 = 0.7205 - 1.$$

Or, since $-.36 = .64 - 1$, then we may write

$$0.0805 - .36 = 0.0805 + .64 - 1 = 0.7205 - 1.$$

Example 4. Find the value of

$$x = \left[\frac{(12.67)^{\frac{3}{4}} \times \sqrt[3]{0.06872}}{(1.235)^2 \times 0.2679}\right]^3.$$

Solution. By Theorems I, II, III, and IV,

$$\log x = 3[\tfrac{3}{4} \log 12.67 + \tfrac{1}{3} \log 0.06872 - (2 \log 1.235 + \log 0.2679)].$$

$$\begin{aligned}\tfrac{3}{4} \log 12.67 &= \tfrac{3}{4}(1.1028) &&= 0.8271\\ \tfrac{1}{3} \log 0.06872 &= \tfrac{1}{3}(1.8371 - 3) &&= 0.6124 - 1\\ \log N &= &&\overline{1.4395 - 1}\\ 2 \log 1.235 &= 2(0.0917) &&= 0.1834\\ \log 0.2679 &= &&0.4279 - 1\\ \log D &= &&\overline{0.6113 - 1}\end{aligned}$$

Since

$$\log x = 3(\log N - \log D),$$

then

$$\begin{aligned}\log N &= 1.4395 - 1\\ \log D &= 0.6113 - 1\\ \log x &= \overline{3(0.8282)}\\ &= 2.4846\end{aligned}$$

Therefore

$$x = 305.2.$$

The characteristic 2 indicates that x is a number with three digits to the left of the decimal point.

When a logarithm is to be multiplied by a fraction, it is best first to multiply the logarithm by the numerator of the fraction and then to divide the product by the denominator of the fraction. In this manner the error will be kept to a minimum.

Example 5. Find the value of

$$x = \sqrt[5]{\frac{(0.002674)^{3/4}(5.2608)^3}{(3.7426)^2\sqrt{27.35}}}.$$

*Solution.** If we use cologarithms, then

$$\log x = \tfrac{1}{5}[\tfrac{3}{4}\log 0.002674 + 3\log 5.2608 + 2\operatorname{colog} 3.7426 + \tfrac{1}{2}\operatorname{colog} 27.35].$$

$$\begin{aligned}
\log (0.002674)^{3/4} &= \tfrac{3}{4}(0.42716 - 3) &&= 0.07037 - 2\\
\log (5.2608)^3 &= 3(0.72105) &&= 2.16315\\
\operatorname{colog} (3.7426)^2 &= 2(9.42683 - 10) &&= 18.85366 - 20\\
\operatorname{colog} \sqrt{27.35} &= \tfrac{1}{2}(8.56304 - 10) &&= 4.28152 - 5\\
\log x &= \tfrac{1}{5}(25.36870 - 27)\\
&= \tfrac{1}{5}(3.36870 - 5)\\
&= 0.67364 - 1
\end{aligned}$$

hence,

$$x = 0.47178.$$

Quite frequently it saves time and lessens the chance of error, if an outline of the solution is made first and is filled in later. To illustrate, the outline for the solution of

$$x = \frac{\sqrt[3]{(68.75)^2}(0.278)^3}{\sqrt{0.005725}}$$

follows and should be filled in by the student.

$$\log x = \tfrac{2}{3}\log 68.75 + 3\log 0.278 + \tfrac{1}{2}\operatorname{colog} 0.005725.$$

$$\begin{aligned}
\log \sqrt[3]{(68.75)^2} &= \tfrac{2}{3}(\qquad) &&=\\
\log (0.278)^3 &= 3(\qquad) &&=\\
\operatorname{colog} \sqrt{0.005725} &= \tfrac{1}{2}(\qquad) &&= ____\\
\log x &= \\
x &= 4.765
\end{aligned}$$

* Calculations are made by using a five-place table of logarithms.

Negative numbers do not have any real logarithms. In case a problem contains the product, quotient, and powers of negative numbers, find by logarithms the numerical value of the expression, considering all numbers as positive, and then attach the proper sign to the result. The sign of the result is determined independent of logarithms. To illustrate:

$$\frac{a(-b)^3(-c)^4}{\sqrt[5]{-d}\,\sqrt{e}} = \frac{(-1)^3(-1)^4}{\sqrt[5]{-1}} \times \frac{ab^3c^4}{\sqrt[5]{d}\,\sqrt{e}}. \tag{32}$$

The value of $\dfrac{ab^3c^4}{\sqrt[5]{d}\,\sqrt{e}}$ can be found by logarithms, and since

$$\frac{(-1)^3(-1)^4}{\sqrt[5]{-1}} = \frac{(-1)^7}{-1} = +1, \tag{33}$$

the answer must be positive.

EXERCISES

By means of logarithms, find the value, correct to four figures, of each of the following fifty-two expressions:

1. $(6{,}247)(7.887)$.

2. $(0.5673)^{\frac{2}{3}}$.

3. $\sqrt[7]{0.00075}$.

4. $\sqrt[3]{6{,}789}$.

5. $(3{,}570)(0.00672)$.

6. $(-768.2)(-0.75)^2$.

7. $\sqrt[5]{0.005687}$.

8. $\sqrt[4]{(0.006527)^3}$.

9. $\dfrac{4{,}216}{2{,}745}$.

10. $\dfrac{54.75}{(1.006)^3}$.

11. $\dfrac{27.48}{677.8}$.

12. $\sqrt[5]{\dfrac{(-4{,}682{,}000)^3}{(20{,}300)^4}}$.

13. $\dfrac{\sqrt{74}}{\sqrt[3]{128}}$.

14. $\dfrac{2{,}856\,\sqrt[5]{(28.04)^2}}{(1{,}005)^3(0.065)^4}$.

15. $\dfrac{2 \cdot 3 \cdot 7 \cdot 19 \cdot 23 \cdot 37}{5 \cdot 11 \cdot 13 \cdot 17 \cdot 29 \cdot 31}$.

16. $\dfrac{1}{28.75}\sqrt{\dfrac{2{,}684}{0.0909}}$.

17. $\dfrac{1}{5.875}$.

18. $\dfrac{(26.87)^{\frac{3}{4}}(0.909)^5}{\sqrt{707^3(0.0002003)}}$.

19. $\left(\dfrac{62.78}{0.0672}\right)^{\frac{3}{4}}$.

20. $\dfrac{2.235 \times 10^{-11}}{(0.000687)^3}$.

21. $\sqrt[4]{\dfrac{(6.789)^3}{0.2756}}$.

22. $\dfrac{\sqrt[4]{629}\ (0.00901)^{\frac{2}{3}}}{(28.75)^3\ \sqrt[5]{0.0909}}$.

23. $(1.06)^{10}$.

24. $(0.01)^{\frac{3}{4}}$.

25. $\dfrac{(367.2)^2\sqrt[3]{1.234}}{(0.00035)(27.64)^{5/3}}$.

26. $\dfrac{(24.68)^{-4}(0.0607)^{3/4}}{\sqrt{(90.07)^{-5}(0.010)^{4/3}}}$.

27. $\sqrt[3]{\dfrac{(52{,}750)^{2/3}}{(68{,}740)^3(0.00076)^4}}$.

28. $\dfrac{(0.0256)^{3/4}(678.46)^4\sqrt[5]{(2.6847)^3}}{[4.0004\sqrt[3]{11.111}]^4(98.989)^5}$.

29. $\left[\dfrac{1.8(6.7896)^{3/2}}{\sqrt[4]{19{,}257}}\right]^5$.

30. $\dfrac{(\frac{1\,1\,1}{1\,3})^{3/4}\sqrt[3]{\frac{9\,1}{2\,6}}(26\frac{3}{4})^2}{(68\frac{2}{3})^3(\frac{2\,6}{2\,5})^2(\frac{4}{5})^7}$.

31. $\dfrac{\sqrt[7]{(268.74)^3}\sqrt[3]{0.00024}(60.007)^{3/4}}{\sqrt[5]{62{,}807}(0.002705)^{4/5}(\sqrt[3]{8.75})^5}$.

32. $\dfrac{(5{,}784-2{,}786)^2}{(1{,}867+1{,}324)^3}$.

33. $\sqrt{3{,}476^2+8{,}679^2}$.

34. $\sqrt{(6{,}784)^2+(5{,}726.8)^2}$.

35. $\sqrt{(2{,}654)^2-(1{,}892)^2}$. Hint, $\sqrt{a^2-b^2}=\sqrt{(a-b)(a+b)}$.

36. $\sqrt[5]{64.809}+(0.0098)^{-3/4}$.

37. $68^{1.2}$.

38. $\dfrac{1}{28^{0.23}}$.

39. $0.075^{0.35}$.

40. $0.037^{-0.4}$.

41. $e^{2.5}$.

42. $e^{1.8}$.

43. $e^{-0.2}$.

44. $e^{-0.24}$.

45. $2.68^{-1.37}$.

46. $\sqrt[3]{\log 0.006}$.

47. $0.012^{0.12}$.

48. $(\log 2.0087)^{0.036}$.

49. $(0.368)^3-\log 0.0285$.

50. $\log 16.67-\sqrt[4]{456.98}$.

51. $[(2.387)^{-2.34}+(0.06)^{-0.7}]^{3/4}$.

52. $\dfrac{(34.67)^{-2.3}-0.0076}{0.5632}$.

53. The area of a circle is $A=\pi r^2$. Find the area of a circle of radius 6,254 feet ($\pi=3.1416$).

54. The volume of a sphere is $V=\frac{4}{3}\pi r^3$. Find the radius of a sphere whose volume is 35,240 cubic feet.

55. If money is compounded annually, the amount A of P dollars in n years with interest at i per cent is $A=P(1+i)^n$. Find the amount of \$785.70 at 6% for 15 years. What principal will amount to \$2,000 at 5% in 12 years?

56. If money is compounded annually, the amount A of P dollars in n years with interest at i per cent is $A=P(1+i)^n$. Find the amount of \$1,475.50 at $5\frac{1}{2}$% for 15 years.

57. The amount of an annuity of R dollars per year for n years at a rate of interest i compounded annually is $s_{\overline{n}|}=R\dfrac{(1+i)^n-1}{i}$. Find the amount of an annuity of \$500 for 20 years at 4%. At 6%.

58. The present value of an annuity of R dollars per year for n years at a rate of interest i compounded annually is $a_{\overline{n}|}=R\dfrac{1-(1+i)^{-n}}{i}$.

Find the present value of an annuity of $475.25 for 20 years at $4\frac{1}{2}\%$. At $5\frac{1}{2}\%$.

59. The area of a triangle with sides a, b, and c is given by the formula $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $2s = a + b + c$. Find the area of a triangle with sides 6,245 feet, 13,445 feet, and 11,738 feet in length.

60. Given $A = P(1+i)^n$. How long will it take for a given principal to double itself at 5%? At 6%?

61. The amount of an annuity of R dollars per year payable in p installments per year for n years at a rate of interest i compounded m times a year is $s_{\overline{n}|}^{p} = \dfrac{R}{p}\dfrac{\left(1+\frac{i}{m}\right)^{mn}-1}{\left(1+\frac{i}{m}\right)^{m/p}-1}$. Find the amount of an annuity of $1,000 paid in 12 equal monthly installments for 10 years if money is worth 6% compounded semiannually.

62. The present value of an annuity of R dollars per year payable in p installments per year for n years at a rate of interest i compounded m times a year is $a_{\overline{n}|}^{p} = \dfrac{R}{p}\dfrac{1-\left(1+\frac{i}{m}\right)^{-mn}}{\left(1+\frac{i}{m}\right)^{m/p}-1}$. Find the present value of an annuity of $1,000 paid in 12 equal monthly installments for 15 years if money is worth 5% compounded quarterly.

63. The stretch S of a wire of length l and radius r by a weight m is $S = \dfrac{mgl}{\pi r^2 k}$, where k is a constant for a given material and is called Young's modulus. Find how much a steel wire of length 172 centimeters and radius 0.032 centimeter will be stretched by a weight of 8,290 grams ($g = 980$, and k for steel is 22×10^{11}).

64. Find the length l of a copper wire with diameter 0.5 centimeter which is stretched 1.75 centimeters by a weight of 10,200 grams (k for copper is 12×10^{11}). (See Exercise 63.)

65. The time T in seconds of a single swing of a pendulum of length l in centimeters is $T = \pi\sqrt{\dfrac{l}{g}}$. Find the length of a seconds pendulum in a locality where $g = 980$ centimeters.

66. The number N of vibrations per second made by a string of length l, radius r, density d, stretched by a weight m is $N = \dfrac{1}{2l}\sqrt{\dfrac{mg}{\pi r^2 d}}$. Find the stretching weight necessary to make a nickel wire 50 centimeters long with radius 0.25 centimeter make 150 complete vibrations per second (density of nickel is 8.9).

67. If money is compounded m times a year, the amount of P dollars in n years at i per cent is $A = P\left(1 + \frac{i}{m}\right)^{mn}$. Find the amount of \$1,000 at 6% for 10 years if money is compounded (a) semiannually, (b) quarterly, (c) monthly.

68. The weight W in pounds that will crush a solid cylindrical cast-iron column d inches in diameter and l feet long is $W = 98{,}920\,\frac{d^{3.55}}{l^{1.7}}$. (a) Find the crushing load for a rod 5 feet long and 5.2 inches in diameter. (b) What length of rod 5 inches in diameter will support 40 tons?

69. The friction drop in pressure p in inches per foot is

$$p = 0.000822\,\frac{v^{1.77}}{d^{1.275}},$$

where v is the velocity of water in inches per second and d is the inside diameter of the pipe in inches. Find p if $v = 150$ feet per minute and $d = \frac{3}{4}$ inch.

70. The lift L in pounds for an airplane is $L = 0.0012 C_l A V^2$, where C_l is the coefficient of lift, A is the area of the wings in square feet, and V is the velocity of the wind in feet per second. (a) Find the lifting load of a plane with a wing area of 745 square feet, if the wind velocity is 350 miles per hour, and if $C_l = .59$. (b) What wind speed is necessary for a plane with a wing area of 1,275 square feet to lift 57 tons if $C_l = .67$?

71. The horsepower needed to move a plane with wing area A square feet at velocity V feet per second is $H.P. = .0000022 C_d A V^3$, where C_d is the coefficient of drag. Find the horsepower necessary to fly the planes in problem 70, parts (a) and (b), if $C_d = .028$.

72. A boiler of horsepower H.P. should have a chimney h feet high and an inside cross-sectional area of A square feet, determined by the relation $\text{H.P.} = 3.33(A - 0.6\sqrt{A})\sqrt{h}$. What must be the height of a chimney with a cross-sectional area of 12.8 square feet for a boiler of 250 horsepower?

73. The current i flowing in a series circuit of R ohms resistance and L henrys self-inductance t seconds after its source of electromotive force is short-circuited is given by $i = Ie^{-\frac{Rt}{L}}$, where I was the current flowing in the circuit before the short circuit. What self-inductance is necessary to make the current decrease to one-half its value in one-third of a second if $R = .1$ ohm?

74. The rate Q of discharge of water in cubic feet per second over a rectangular weir b feet in width due to a head of H feet of water is $Q = \frac{2}{3} b\sqrt{2}\,gH^{3/2}$. Find Q if $b = 17$ feet, $H = 7$ feet 5 inches, and $g = 32.2$.

75. The relation between absolute temperatures T and pressures P for an adiabatic change in gas from an initial condition to a final condition is $\frac{T}{T_1} = \left(\frac{P}{P_1}\right)^{(n-1)/n}$. For air, $n = 1.3$. (a) What will be the change in temperature of air at 70° F., if the pressure is changed from 15 pounds to 85 pounds (adiabatically)? (b) What increase in pressure is necessary to change the temperature of air under 85 pounds pressure from 65° F. to 75° F., (adiabatically)? The zero of the absolute temperature scale is about 460° below the zero of the Fahrenheit scale; that is, $T = 460° +$ °F.

Find the values of each of the following formulas (Exercises 76 through 82) for the data given.

76. $s = vt + \frac{1}{2}gt^2$. Find s when $v = 175$, $t = 13.5$, and $g = 32.2$.

77. $S = 2\pi rh + 2\pi r^2$. Find S when $r = 28.5$, $h = 33.45$, and $\pi = 3.1416$.

78. $M = \frac{3WDL^2}{4}$. Find M when $W = 350$, $D = 0.035$, and $L = 785$.

79. $T = \frac{3W(A - a)}{6{,}400B}$. Find T when $W = 6{,}840$, $A = 0.68$, $a = 0.24$, and $B = 3.5$.

80. $G = \frac{R}{R^2 + X^2}$. Find G when $R = 0.02675$ and $X = 0.0369$.

81. $N = \frac{1}{2l}\sqrt{\frac{Mg}{m}}$. Find N when $l = 95$, $M = 8{,}750$, $m = 0.076$, and $g = 980$.

82. $z = \frac{t(3kd - 2t)}{3(2kd - t)}$. Find z when $t = 8.4$, $k = 65.4$, and $d = 0.03$.

35. EXPONENTIAL EQUATIONS AND CHANGE OF BASE

Suppose we wish to find what power of a number will equal a given number. This type of problem can be solved by logarithms. To illustrate:

Example 1. Find what power of 2 equals 5.

Solution. In this case we have to solve the equation

$$2^x = 5$$

for x. This can be done by taking the common logarithm of both members of the equation. Then, by Equation (18), Article 30,

$$\log 2^x = x \cdot \log 2 = \log 5.$$

Solving for x

$$x = \frac{\log 5}{\log 2} = \frac{0.6990}{0.3010} = 2.322.$$

Now suppose we wish to find $\log_2 5$. If we let $\log_2 5 = x$, then $2^x = 5$ and the solution of this equation is the same as the solution given in Example 1. Thus we see that the two problems, solving the equation $2^x = 5$ for x and finding the value of x in $x = \log_2 5$, are solved in the same manner. Hence,

$$\log_2 5 = 2.322.$$

It is possible to find the logarithm of a given number N to any base a, provided we know the logarithms of numbers to some given base, say base b. Thus, suppose we wish to find the value of $\log_a N$. From the definition given in Equation (1), Article 30,

$$a^{\log_a N} = N. \tag{34}$$

Then, by taking the logarithm to the base b of each member of Equation (34), we have

$$\log_b (a^{\log_a N}) = (\log_a N)(\log_b a) = \log_b N.$$

Solving this equation for $\log_a N$, we have

$$\log_a N = \frac{\log_b N}{\log_b a}. \tag{35}$$

The logarithm of a number N *to the base* a *is equal to the logarithm of* N *to the base* b, *divided by the logarithm of* a *to the base* b. (Observe that this is the quotient of two logarithms, and hence the two logarithms must be divided and not subtracted.)

Since common logarithms are used to carry out most calculations, we may write

$$\log_a N = \frac{\log_{10} N}{\log_{10} a}. \tag{36}$$

The logarithm of a number N *to any base* a *is equal to the logarithm of* N *to the base* 10, *divided by the logarithm of* a *to the base* 10.

If the ratio given in Equation (36) involves logarithms of positive numbers less than 1, then it is best to express the logarithms of the numbers less than 1 as negative numbers. To illustrate:

Example 2. Find the value of $\log_2 0.05$.

Solution. By Equation (36)

$$\log_2 0.05 = \frac{\log 0.05}{\log 2} = \frac{0.6990 - 2}{0.3010}$$

$$= \frac{-(1.3010)}{0.3010} = -(4.32)$$

Observe that

$$0.6990 - 2 = -2 + 0.6990 = -(1.3010).$$

The equation $2^x = 0.05$ can be solved for x in the same manner since $x \log 2 = \log 0.05$, and thus $x = \dfrac{\log 0.05}{\log 2}$.

EXERCISES

Find the logarithms of the following numbers (Exercises 1 through 12).

1. 5 to the base 2.
2. 125 to the base 1000.
3. 2 to the base 5.
4. 0.07 to the base 100.
5. 100 to the base 6.
6. 50 to the base 0.1.
7. 64 to the base 100.
8. 0.04 to the base 0.02.
9. 8 to the base e.
10. 20 to the base e.
11. 0.14 to the base e.
12. 0.026 to the base e.

Solve the equations in Exercises 13 through 30 for x.

13. $7^x = 2$.
14. $10^{3x} = 678$.
15. $3^{2x-1} = 4$.
16. $12^x = 0.4$.
17. $2^{3x-1} = 4^{2x+1}$.
18. $5^{2x-1} = 125$.
19. $4^{3x+2} = (\frac{1}{2})^{x-5}$.
20. $10^{2x} = 0.0375$.
21. $3^x = \frac{1}{9}$.
22. $0.01^x = 0.02$.
23. $e^x = 13.7$.
24. $e^{2x} = 0.056$.
25. $e^{-x} = 0.045$.
26. $e^{-3x} = 1.056$.
27. $\log_e x = 1.6525$.
28. $\log_e x = 4.2485$.
29. $\log_e x = 5.8693$.
30. $\log_e x = 9.384 - 10$.

36. HOW LOGARITHMS ARE COMPUTED

The values of the natural logarithms are computed from the following series (see Exercise 48, Chapter XXI).

$$\log_e (N + 1) = \log_e N + 2\left[\frac{1}{2N+1} + \frac{1}{3(2N+1)^3} + \frac{1}{5(2N+1)^5} + \cdots\right]. \quad (37)$$

Enough terms can be used to compute the natural logarithms correctly to as many decimal places as may be desired. The natural logarithm of any number $N + 1$ can be computed, provided the value of the natural logarithm of N is known. Since

$\log_e 1$ is zero, we can compute the value of $\log_e 2$ by letting $N = 1$. Thus,

$$\log_e 2 = \log_e 1 + 2\left[\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \frac{1}{9 \cdot 3^9} + \frac{1}{11 \cdot 3^{11}} \cdots\right]$$

$$= 0 + 2(0.3333333 + 0.0123457 + 0.0008230 + 0.0000653 + 0.0000056 + 0.0000005) \tag{38}$$

$$= 2(0.3465734)$$

$$= 0.6931468.*$$

The first six terms of the series are sufficient to compute $\log_e 2$ correctly to six decimal places, but to determine $\log_e 2$ correct to seven decimal places, it would be necessary to use more than six terms. For very large values of N, only a few terms of Equation (37) are needed to calculate $\log_e N$.

The common logarithms are computed from the natural logarithms by means of the relation given in Equation (35). Therefore,

$$\log_{10} N = \frac{\log_e N}{\log_e 10}. \tag{39}$$

Since

$$\log_e 10 = 2.30259 \quad \text{and} \quad \frac{1}{2.30259} = 0.43429448 \cdots,$$

we may write

$$\log_{10} N = \frac{\log_e N}{2.30259}, \tag{40}$$

or

$$\log_{10} N = 0.43429 \log_e N. \tag{41}$$

The common logarithm of any number N *is found, either by dividing the natural logarithm of* N *by* 2.30259, *or by multiplying the natural logarithm of* N *by* $0.43429448 \cdots$.

* To ten decimal places, $\log 2 = 0.6931471806$.

Example. Find the common logarithm of 2.

Solution.

$$\log_{10} 2 = 0.43429 \log_e 2$$
$$= (0.43429)(0.69315) \quad \text{(see Equation 38)}$$
$$= 0.30103.$$

EXERCISES

1. By means of Equation (37), compute the value of each of the following: (a) $\log_e 2$; (b) $\log_e 3$; (c) $\log_e 5$; (d) $\log_e 7$.

2. Is it necessary to use Equation (37) to compute $\log_e 4$? $\log_e 6$?

3. From the results in Exercise 1, compute (a) $\log_e 4$; (b) $\log_e 6$; (c) $\log_e 8$; (d) $\log_e 9$; (e) $\log_e 10$.

4. From the results in Exercise 1, compute the common logarithms of 2, 3, 4, 5, 6, 7, 8, 9, and 10.

5. Given $\log_e 10 = 2.30259$. Find (a) $\log_e 100$; (b) $\log 0.1$; (c) $\log 0.01$.

6. Given $\log_e 2 = 0.69315$ and $\log_e 10 = 2.30259$. Find (a) $\log_e 20$; (b) $\log_e 200$; (c) $\log 0.2$; (d) $\log 0.02$.

In an attempt to simplify the process of multiplication, it was observed by Archimedes as early as two centuries before Christ that multiplication could be made to depend on addition if the powers of numbers were known. Bürgi (1552–1632), a Swiss, was the first to compute a table for this purpose, but this was not published until 1620. John Napier (1550–1617), a Scotch mathematician, computed and published, in 1614, a table of what he called logarithms, and for this reason we say logarithms were invented by Napier. However, it was Henry Briggs (1556–1631), an Englishman, who introduced, in 1617, the first table of common logarithms. This was done partially at the suggestion of Napier a few years before his death.

The idea of logarithms grew out of the idea of arithmetic and geometric progressions. Thus, consider the arithmetic progression

$$0, \quad d, \quad 2d, \quad 3d, \quad 4d, \quad 5d, \quad \cdots$$

and the geometric progression

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad ar^5, \quad \cdots .$$

The terms of the arithmetic progression can be taken as the logarithms of the corresponding terms of the geometric progression. For example, $\log ar^3 = 3d$. Bürgi used $d = 10$, $a = 100{,}000{,}000$, and $r = 1.0001$. Napier used $d = 1.00000005$, $a = 10{,}000{,}000$, and $r = 0.9999999$. For logarithms based on these progressions,

$\log (ar^2)(ar^3) = \log a(ar^5)$, which does not equal $\log ar^2 + \log ar^3 = 2d + 3d = 5d$ unless $a = 1$. Thus, the logarithm of a product does not equal the sum of the logarithms unless $a = 1$. However, at the time Napier invented his logarithms, trigonometric computations were carried on by the aid of proportions such as $\frac{\sin A}{a} = \frac{\sin B}{b}$. From the relation $\log \frac{ar^x \cdot ar^y}{ar^z} = \log ar^{x+y-z}$, we see that the relation $\log \frac{A \cdot B}{C} = \log A + \log B - \log C$ held for his table. In Napier's system of logarithms, $\log M \cdot N = \log M + \log N - \log 1$, and $\log \frac{M}{N} = \log M - \log N + \log 1$. Napier suggested to Briggs that he make $a = 1$, and Briggs decided to let $d = 1$ and $r = 10$; that is, Briggs used the following progressions:

$$\begin{array}{cccccc} 0, & 1, & 2, & 3, & 4, & 5, \\ 1, & 10, & 10^2, & 10^3, & 10^4, & 10^5, \end{array}$$

and so on. Neither Napier nor Briggs thought of logarithms in terms of our modern idea of base. The earlier tables were computed by taking the square root successively. This was done as many as 54 times in succession; that is, $\log 10 = 1$, $\log \sqrt{10} = 0.5$, $\log \sqrt[4]{10} = 0.25$, and so forth. It was not until the eighteenth century that logarithms were computed from series. Briggs, in 1624, introduced the term **characteristic.** The term **mantissa** had been used for the decimal part of any number, and was not generally used to denote a part of the logarithm until 1748 when Euler used it as such. The logarithms to the base e are called **natural logarithms,** or sometimes Napier's logarithms. Although they are not the original Napier's logarithms, there is a fixed relation between the two:

$$\text{Napier } \log N = K \log_e \frac{10^7}{N},$$

$$\text{where } K = -\frac{1.00000005}{\log_e 0.9999999}.$$

CHAPTER VI

Functions and Graphs

37. GRAPHS

Every day we have experiences in which one quantity is changing with respect to a second quantity, such as the change in temperature with respect to time. Suppose the temperature has been observed and recorded for every hour of a given day; then from these observations it is possible to determine (approximately) the change in temperature during a given time of the day, for what hours of the day the temperature was falling, and the highest and lowest temperatures for this day. However, these properties can be more readily seen if the results are exhibited in the form of a graph.

A complete set of observed recordings is called **data.** The accompanying table gives a set of data taken by observing the temperature at every hour of the day for January 30, 1942.

Time	Temp., degrees	Time	Temp., degrees	Time	Temp., degrees
12 p.m.	13	8 a.m.	9	4 p.m.	30
1 a.m.	12.6	9 a.m.	13	5 p.m.	29
2 a.m.	12.2	10 a.m.	20	6 p.m.	27
3 a.m.	12	11 a.m.	24	7 p.m.	26
4 a.m.	11	12 noon	30	8 p.m.	28
5 a.m.	10	1 p.m.	27.8	9 p.m.	27
6 a.m.	10	2 p.m.	29.5	10 p.m.	26
7 a.m.	9.5	3 p.m.	30	11 p.m.	25

To make a graph of these observations, we use two perpendicular lines as reference lines. On the horizontal line mark off a series of points equally spaced and label them the hours of the day as in Fig. 2. On the vertical line mark off a series of points equally spaced to represent the temperature. (In problems of this nature,

the units used on the two reference lines need not be the same in length.) Above each division representing the hours of the day place a point as many units above the horizontal line as the number of units representing the temperature at that hour, and then draw a smooth curve through these points. This smooth curve is called the graph.

If the temperature had been observed more frequently, say every 30 minutes, or every 10 minutes, or every minute, or every second, then the resulting graph would be more nearly the true

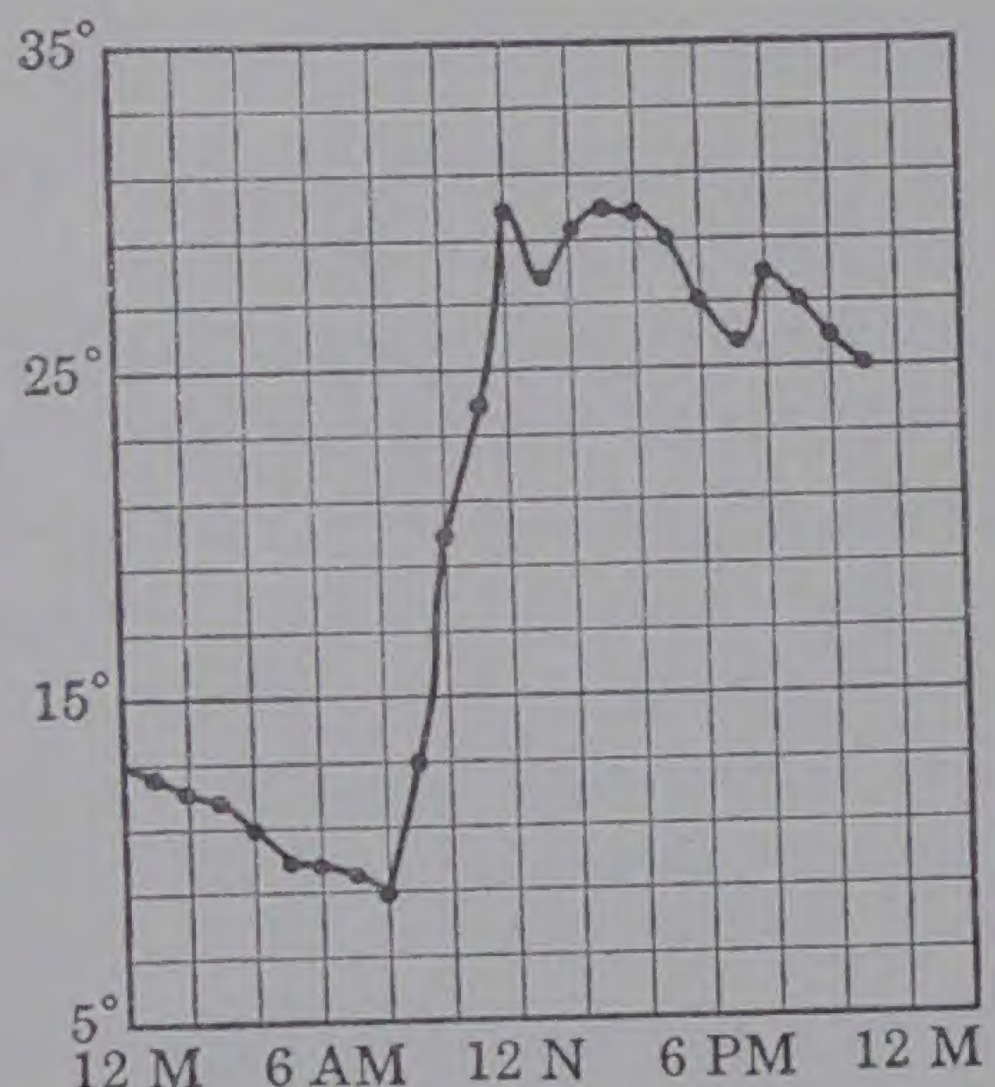


Fig. 2.

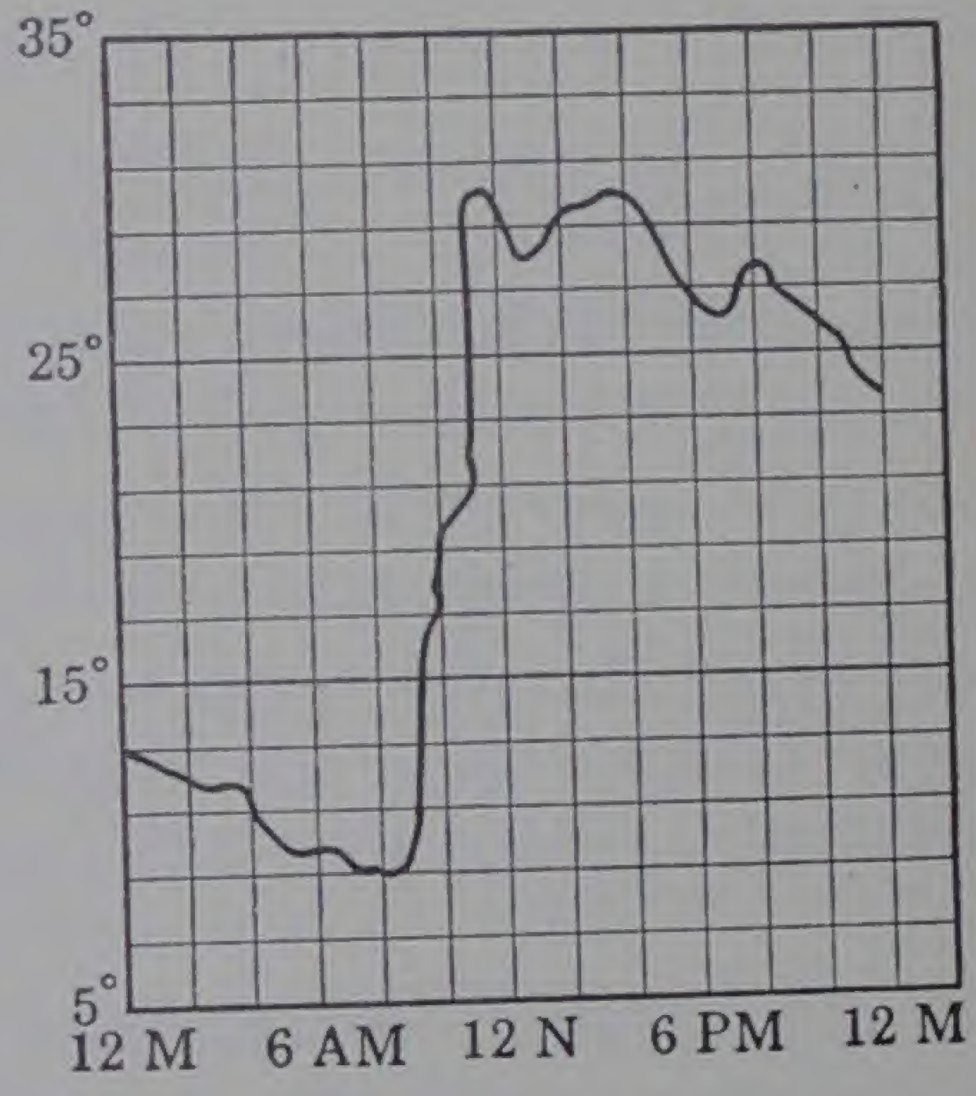


Fig. 3.

graph. In Fig. 3 is given the continuous graph as was recorded by a thermometrograph for the same day.

Since there is no law governing the relation between temperature and time, it is impossible to extend the graph beyond the observed data with any degree of accuracy. However, the temperature for any time between two consecutive hours can be determined from the graph in Fig. 2 with a considerable degree of accuracy. To illustrate, the temperature at 8:30 a.m. was approximately 10° above zero; at 3:30 p.m. the temperature was approximately 30° above zero; and so on.

In the case of uniform motion, if the distance is observed at regular intervals of time and is plotted, it is found that the graph is a straight line. This graph can be extended beyond the observed data, and from the extended graph, distances can be determined accurately for any time. To illustrate, suppose a man walks at the uniform rate of 4 miles per hour and the distance is observed for every hour; the following data will result:

Time	0	1	2	3	4	5	6
Miles	0	4	8	12	16	20	24

The graph of these is given in Fig. 4. If the graph were extended, we would find the distance for 8 hours to be 32 miles, the distance for 10 hours to be 40 miles, and so on. In this case, the relation connecting distance S and time T is

$$S = 4T.$$

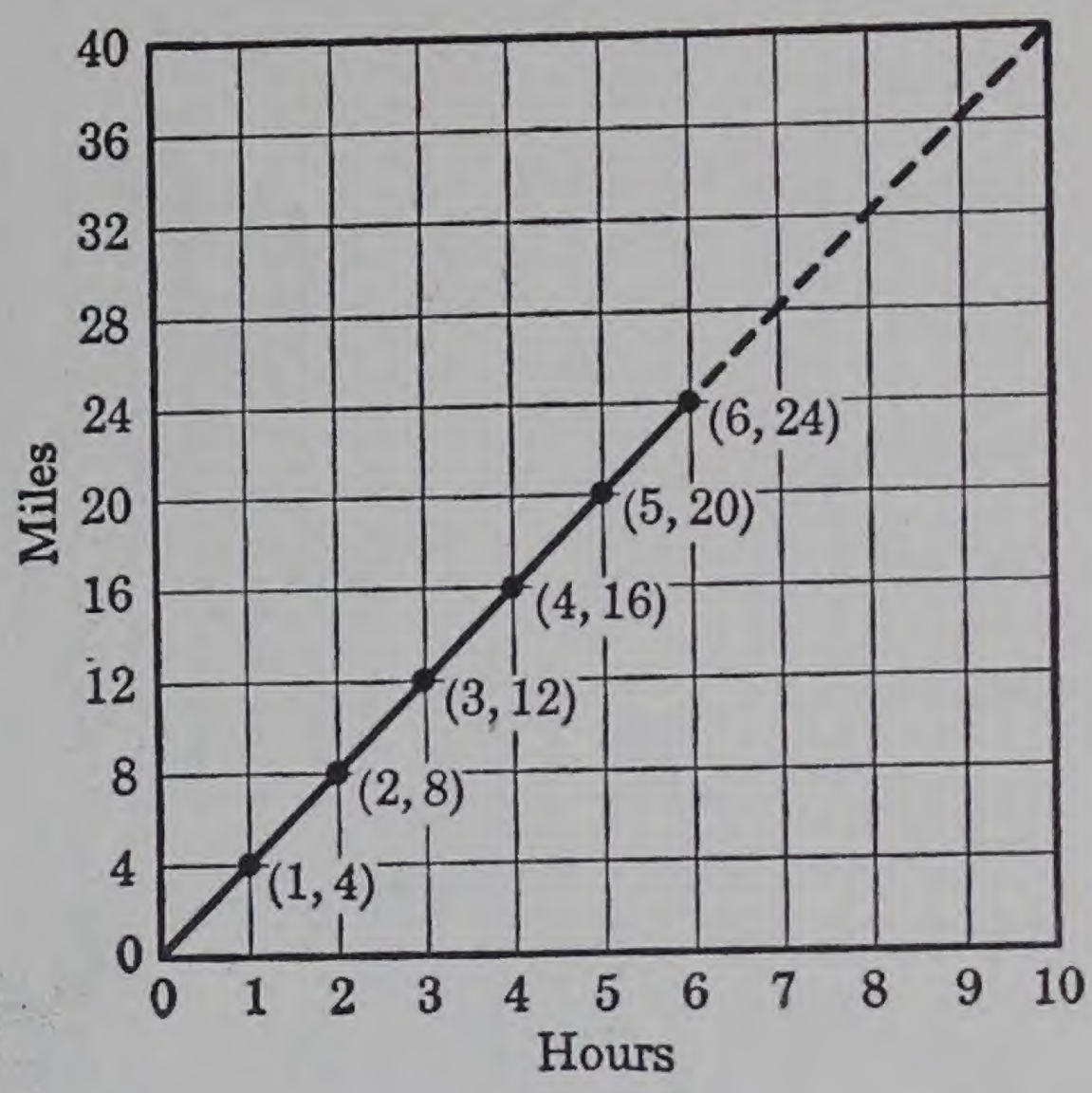


FIG. 4.

When graphing observed data, lay off on the horizontal line the arbitrarily chosen uniform part of the data, and let the observed data be the measure of the distance from the horizontal line.

Sometimes a graphic representation of certain kinds of data in economics is best given by means of rectangles. For example, the graphic representation in Fig. 5 (page 102) shows the production of automobiles over a period of time, the number of units of area in a given rectangle representing the number of automobiles produced for that year. In examples of this type, if points were plotted and a smooth curve was drawn through the plotted points, the graph would be practically meaningless between two successive plotted points because the graph does not show the distribution of production throughout the year but merely gives the production

Year	Production	Year	Production	Year	Production
1915	9.7×10^5	1924	36.0×10^5	1933	19.2×10^5
1916	16.2×10^5	1925	42.7×10^5	1934	27.5×10^5
1917	18.7×10^5	1926	43.0×10^5	1935	39.5×10^5
1918	11.7×10^5	1927	34.0×10^5	1936	44.5×10^5
1919	18.8×10^5	1928	43.6×10^5	1937	48.1×10^5
1920	22.3×10^5	1929	54.6×10^5	1938	24.9×10^5
1921	16.2×10^5	1930	33.6×10^5	1939	35.8×10^5
1922	25.4×10^5	1931	23.9×10^5	1940	44.7×10^5
1923	40.3×10^5	1932	13.7×10^5	1941	48.4×10^5

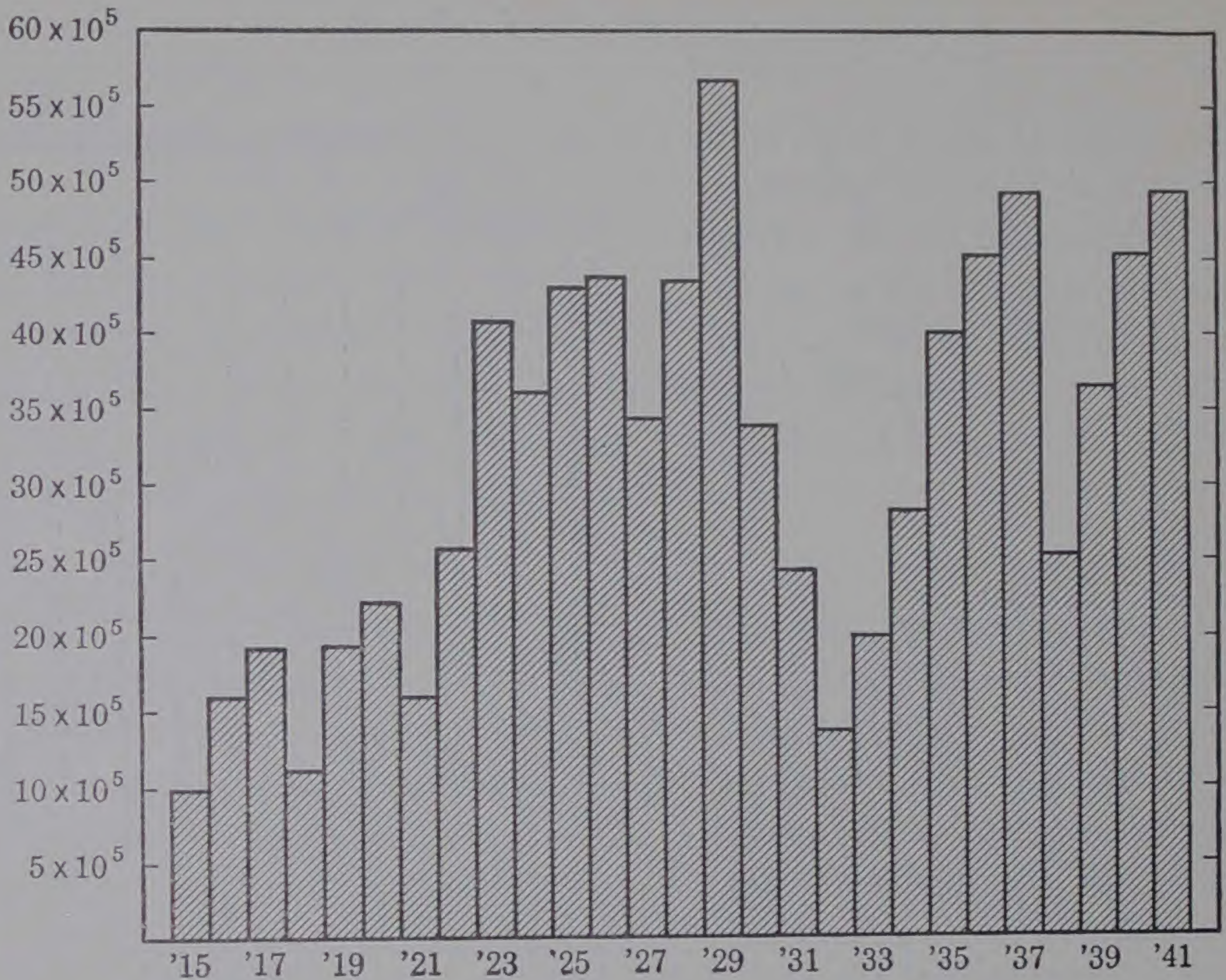


FIG. 5.

for the year. The yearly production is shown in the table at the foot of page 101.*

EXERCISES

Graph each of the following sets of data (Exercises 1 through 8) to a suitable chosen scale.

1. The population of the United States at each census from 1790 to 1920 is given by the following table.* (a) Estimate from the graph the

Year	Population	Year	Population	Year	Population
1790	3.9×10^6	1840	17.1×10^6	1890	62.9×10^6
1800	5.3×10^6	1850	23.9×10^6	1900	76.0×10^6
1810	7.2×10^6	1860	31.4×10^6	1910	92.0×10^6
1820	9.6×10^6	1870	39.8×10^6	1920	105.7×10^6
1830	12.9×10^6	1880	50.2×10^6	1930	?

population in 1845, in 1868, in 1907. (b) When was the population 10,000,000, 100,000,000, 110,000,000? (c) Extend the graph to estimate the 1930 census. (d) Predict the 1940 census. (e) When will the population be 130,000,000?

2. Graph the following sets of data on the same graph. One part of the table* gives the number of automobiles registered yearly in the

* Taken from the Statistical Abstract of the United States in 1946.

United States from 1916 to 1935, the other part of the table gives the number of deaths due to automobiles during the same period. Use different units for registration and deaths.

Year	Registration	Deaths	Year	Registration	Deaths
1916	2.51×10^6	4.7×10^3	1926	19.84×10^6	20.6×10^3
1917	3.53×10^6	6.1×10^3	1927	21.08×10^6	23.1×10^3
1918	4.72×10^6	7.3×10^3	1928	23.13×10^6	26.3×10^3
1919	5.92×10^6	7.8×10^3	1929	25.10×10^6	29.4×10^3
1920	7.44×10^6	8.9×10^3	1930	25.35×10^6	31.2×10^3
1921	8.58×10^6	9.9×10^3	1931	24.71×10^6	32.0×10^3
1922	10.37×10^6	11.5×10^3	1932	23.08×10^6	28.0×10^3
1923	13.36×10^6	14.2×10^3	1933	24.10×10^6	31.4×10^3
1924	15.71×10^6	15.2×10^3	1934	25.21×10^6	36.1×10^3
1925	17.87×10^6	17.1×10^3	1935	26.51×10^6	36.4×10^3

Extend the graph to estimate the number of registered automobiles and the number of deaths due to automobiles in 1936, 1937, and 1940.

3. The following table* gives the rate of deaths due to automobiles in rural districts (under 10,000 population) and in cities (10,000 or more population) per 100,000 inhabitants. Plot on the same graph. Predict the death rates for 1930 and 1931.

Year	Rural	Cities	Year	Rural	Cities	Year	Rural	Cities
1915	3.9	7.7	1920	6.2	14.7	1925	12.1	22.4
1916	4.8	9.7	1921	7.5	15.4	1926	15.8	24.8
1917	5.9	11.9	1922	8.3	16.9	1927	17.4	26.7
1918	5.7	13.1	1923	10.2	19.7	1928	19.0	28.4
1919	5.6	13.5	1924	10.9	20.6	1929	21.5	30.4

4. By means of rectangles, represent graphically the following table* which gives the total number of commercial failures for each year from 1905 to 1931.

Year	Failures	Year	Failures	Year	Failures
1905	11,520	1914	18,280	1923	18,718
1906	10,682	1915	22,156	1924	20,615
1907	11,725	1916	16,993	1925	21,214
1908	15,690	1917	13,855	1926	21,773
1909	12,924	1918	9,982	1927	23,146
1910	12,652	1919	6,451	1928	23,842
1911	13,441	1920	8,881	1929	22,909
1912	15,452	1921	19,652	1930	26,355
1913	16,037	1922	23,676	1931	28,285

* Taken from the Statistical Abstract of the United States in 1946.

5. The American Experience Mortality Table gives the number of persons surviving at various ages out of 100,000 living at age 10. (a) Estimate the number living at age 18, age 21, age 32, age 57, age 64. (b) At what ages are 95,000, 72,000, 55,000, 30,000, 10,000, 100 respectively still living?

Age	Surviving	Age	Surviving	Age	Surviving
10	100,000	40	78,106	70	38,569
15	96,285	45	74,173	75	26,237
20	92,637	50	69,804	80	14,474
25	89,032	55	64,563	85	5,485
30	85,441	60	57,917	90	847
35	81,822	65	49,341	95	3

6. The amount of \$1 placed at compound interest for 10 years at various rates of interest is given in the following table. What is the amount

Rate, %	Amount	Rate, %	Amount	Rate, %	Amount
1	\$1.105	5	\$1.629	9	\$2.368
2	1.219	6	1.791	10	2.594
3	1.344	7	1.967	11	2.839
4	1.480	8	2.159	12	3.105

of \$1 for 10 years at $4\frac{1}{2}\%$, at $5\frac{3}{4}\%$, at $6\frac{1}{4}\%$, at $10\frac{1}{2}\%$? What rate will make \$1 double itself in 10 years? What rate will treble itself in 10 years?

7. The amount of \$1 placed at 5% compounded annually for various years is given in the following table. What is the amount for 4 years?

Years	Amount	Years	Amount	Years	Amount
5	\$1.276	20	\$2.653	35	\$5.516
10	1.629	25	3.386	40	7.040
15	2.079	30	4.322	45	8.985

8 years? 12 years? 28 years? 42 years? How long will it take \$1 to double itself at 5%? treble itself at 5%?

8. Graph the following common logarithms of numbers. From the graph, determine the logarithm of 2.5, 4.5, 15, and 0.7.

Number	Logarithm	Number	Logarithm	Number	Logarithm
0.1	−1.000	4	0.602	8	0.903
1	0.000	5	0.699	9	0.954
2	0.301	6	0.778	10	1.000
3	0.477	7	0.845	20	1.301

38. FUNCTIONS

In physics, chemistry, statistics, economics, and other subjects there are many examples in which there is a definite relation connecting the change of one quantity with the change of a second quantity. To illustrate:

In uniform motion of constant velocity V_0, the relation connecting distance S with time T is

$$S = V_0 T. \tag{1}$$

In the case of freely falling bodies in a vacuum, the relation connecting distance S with time T is

$$S = V_0 T + \tfrac{1}{2} g T^2, \tag{2}$$

where V_0 is the initial velocity and g is the gravity constant, or acceleration ($g = 32.2$ feet per second per second).

In the case of direct current we have Ohm's law which states that the current I through a simple circuit containing an electromotive force E and resistance R is

$$I = \frac{E}{R}. \tag{3}$$

There are many other examples.

If there exists a known relation connecting the different quantities, then a graph can be constructed by computing a set of data from the equation or functional relation connecting the two quantities. To illustrate, in the case of uniform motion of 4 miles per hour, the distance S for any time T can be computed from the relation

$$S = 4T. \tag{4}$$

To do this, assign values to T and solve for the corresponding values of S. Thus, if $T = 2$, $S = 8$; if $T = 3\frac{1}{2}$, $S = 14$; and so on. In this manner all the data given for Fig. 4 can be calculated. The graph is obtained by drawing a smooth curve through the

points obtained by plotting the computed pairs of values of T and S.

In the example for uniform motion

$$S = V_0T, \tag{5}$$

there is considerable difference in the role played by V_0 and that played by S and T. S represents distance, T time, and V_0 the constant velocity. V_0 is called a constant, while S and T are called variables.

A **constant** is a number, or a symbol for a number, that has a definite value throughout a particular discussion. In the foregoing examples, V_0, g, and $\frac{1}{2}$ are constants. Now for a particular discussion V_0 can have any value whatsoever, but throughout this particular discussion V_0 must remain fixed. Likewise g changes its value with change in altitude, but for a given altitude g has a fixed value.

A **variable** is a symbol that is free to change from one value to another value in a particular discussion. In the examples given, S, T, E, I, and R are variables.

In the formula for uniform motion, where the velocity V_0 is constant, the time T can have any value whatsoever, but for a given value of T

$$S = V_0T.$$

Thus, T is spoken of as being the independent variable and S the dependent variable; that is, S depends on T for its value.

However, if we solve for T, then S can have any value whatsoever, but for a given value of S

$$T = \frac{S}{V_0}.$$

In this case, S is said to be the independent variable and T the dependent variable, because T now depends on S for its value.

A variable to which we assign an arbitrary value is called the **independent variable,** and the variable whose value is thus determined is called the **dependent variable.**

When there is a definite relation connecting the two variables, we say that the one variable is a function of the other variable. *If two variables are so related that a given value of the one variable determines one or more values of the second variable, then the second variable is said to be a function of the first variable.*

Leibniz used the term function in 1692, but it was first defined in its present form by the Swiss mathematician Johann Bernoulli (1667–1748), in 1698, in a letter to Leibniz; the letter was published in 1718.

When we wish to state that y is a function of x, without specifying the relation between the variables, we write $y = f(x)$, which is read: "y is a function of x," or for short is read: "f of x." The symbol $f(x)$ was first used in 1734 by Euler. Other symbols, such as $F(x)$, $\phi(x)$, and so forth, may be used to represent functions of the variable x. These symbols are also used at times when the relation connecting the variables is known.

In the preceding illustrations, the function giving the relation of the variables is known, and we may write

$$S = f(T) = V_0T,$$
$$S = f(T) = V_0T + \tfrac{1}{2}gT^2$$
$$I = f(E, R) = \frac{E}{R},$$
$$T = f(S) = \frac{S}{V_0}.$$

The symbol $f(\quad)$ is very useful to denote a certain operation. To illustrate, suppose we have given

$$f(x) = 2x^2 - 3x + 5,$$

then by $f(3)$ is meant the value of $2x^2 - 3x + 5$ when 3 is substituted for x. Thus,

$$f(3) = 2 \cdot 3^2 - 3 \cdot 3 + 5 = 14.$$

Likewise,

$$f(a) = 2a^2 - 3a + 5,$$
$$f(-1) = 2(-1)^2 - 3(-1) + 5 = 10,$$
$$f(x + 1) = 2(x + 1)^2 - 3(x + 1) + 5 = 2x^2 + x + 4,$$
$$f(2x) = 2(2x)^2 - 3(2x) + 5 = 8x^2 - 6x + 5.$$

QUESTIONS

1. Define constant.
2. Define variable.
3. What is meant by independent variable?
4. What is meant by dependent variable?
5. What is meant by a function of two variables?
6. Give the meaning of the symbol $f(x)$ and its use.

EXERCISES

1. If $f(x) = x^2 - 5x + 1$, find $f(1)$, $f(-2)$, $f(0)$, $f(3)$, $f(a)$.

2. If $f(x) = x^3 - 2x^2 + 1$, find $f(-x)$, $f(2x)$, $f(x^2)$, $f\left(\frac{1}{x}\right)$.

3. If $f(x) = 3x - 1$, find $f\left(\frac{1}{2}\right)$, $f\left(\frac{2}{3}\right)$, $f\left(-\frac{3}{5}\right)$, $f(y)$, $f\left(\frac{1}{3}\right)$.

4. If $f(x) = 3x^2 - x + 2$, find $f(x + 2)$, $f(x^2 - 2)$, $f(-x)$, $f\left(\frac{1}{x}\right)$.

5. If $f(x) = \frac{2x + 1}{3x - 1}$, find $f\left(\frac{1}{2}\right)$, $f\left(-\frac{1}{2}\right)$, $f\left(-\frac{1}{3}\right)$, $f\left(\frac{1}{3}\right)$, $f(0)$.

6. If $f(x) = \frac{x^2 + 4}{x - 1}$, find $f(2)$, $f(0)$, $f(1)$, $f(-2)$, $f(-1)$.

7. If $f(t) = 1 + 3t - t^2$, find $f(2)$, $f(-3)$, $f\left(-\frac{1}{2}\right)$, $f\left(\frac{2}{3}\right)$.

8. If $F(x) = \frac{x^2 - 2}{x + 2}$, find $F\left(\frac{3}{4}\right)$, $\frac{F(3)}{F(4)}$, $F(3) \cdot F(4)$, $F(3) + F(4)$.

9. If $f(x) = x^2 - 1$, find $f(2x)$, $f(x^2)$, $f\left(\frac{1}{x}\right)$.

10. If $F(x) = x^2$, find $F(a + b) \cdot F(a - b)$.

11. If $\phi(x) = \frac{x - 3}{x^2 + 2}$, find $\phi(x + 1)$, $\phi(x - 2)$, $\phi\left(\frac{x}{2}\right)$.

12. If $f(x) = x^4 - 2x^2 + x^3 - 1$, and $g(x) = x^3 - 1 + 2x^2$, find $f(x) \cdot g(x)$, $\frac{f(x)}{g(x)}$.

13. If $F(x) = \frac{2x^2 - 3}{4x}$, find $F(-1)$, $F(x - 2)$, $F(x^2)$.

14. If $f(x, y) = x^2 - 3xy - 2y^2$, find $f(2, 3)$, $f(-1, 2)$, $f(-3, -2)$.

15. If $f(x) = \frac{1}{x} + x$, find $f\left(\frac{1}{x}\right)$, $f(x^2)$, $[f(x)]^2$.

16. If $g(x, y) = 3x^2 - 2xy + 4x + 3$, find $g(x + 1, x)$.

17. If $y = f(x) = x + 1$, find $f(y)$, $f(y^2)$, $f\left(\frac{1}{y}\right)$ as functions of x.

18. If $f(x, y) = (3x + 2y)(2x - 5y)$, find $f(2, -3) - f(3, -1)$.

Give the complete function for each of the following (Exercises 19 through 26).

19. The area of a rectangle is a function of its length and breadth.

20. The area of a triangle is a function of its altitude and base.

21. The area of a circle is a function of its radius.

22. The circumference of a circle is a function of its radius.

23. The volume of a sphere is a function of its radius.

24. The surface of a sphere is a function of its radius.

25. The volume of a right circular cone is a function of the radius of its base and the height of the cone.

26. The volume of a right circular cylinder is a function of the radius of its base and the height of the cylinder.

27. The freezing point of pure water is 32° Fahrenheit or 0° Centigrade; the boiling point of water is 212° Fahrenheit or 100° Centigrade. (a) Express Fahrenheit reading as a function of Centigrade. (b) Express Centigrade reading as a function of Fahrenheit. (c) 77° Fahrenheit is how many degrees Centigrade?

39. SYSTEM OF RECTANGULAR COÖRDINATES

For general graphing we use what is called the system of rectangular coördinates. In this system the plane is divided into four parts, called **quadrants,** by two perpendicular lines. The quadrants are called **first, second, third,** and **fourth quadrants,** respectively, as is shown in Fig. 6. The horizontal line $X'X$ is called the ***x*-axis** and the vertical line $Y'Y$ is called the ***y*-axis.** The two lines $X'X$ and $Y'Y$ are called **coördinate axes,** and their intersection O is called the **origin.** On each of the axes, construct a numbered scale with an arbitrary length as a unit and the origin as the zero point of each. The units on the two axes are usually the same; however, this is not necessary. *The scale on each axis should be indicated by numbering points at uniform intervals along each of the axes.* **Positive numbers** are to the **right** on the horizontal line $X'X$, and **upward** on the vertical line $Y'Y$. **Negative numbers** are to the **left** on $X'X$ and **downward** on $Y'Y$. Every point in this plane can now be located with reference to these two perpendicular lines in the following manner. A point P is at a given distance from the y-axis, measured parallel to the x-axis, called the x-distance or the **abscissa,** and is at a given distance from the x-axis, measured parallel to the y-axis, called the y-distance or

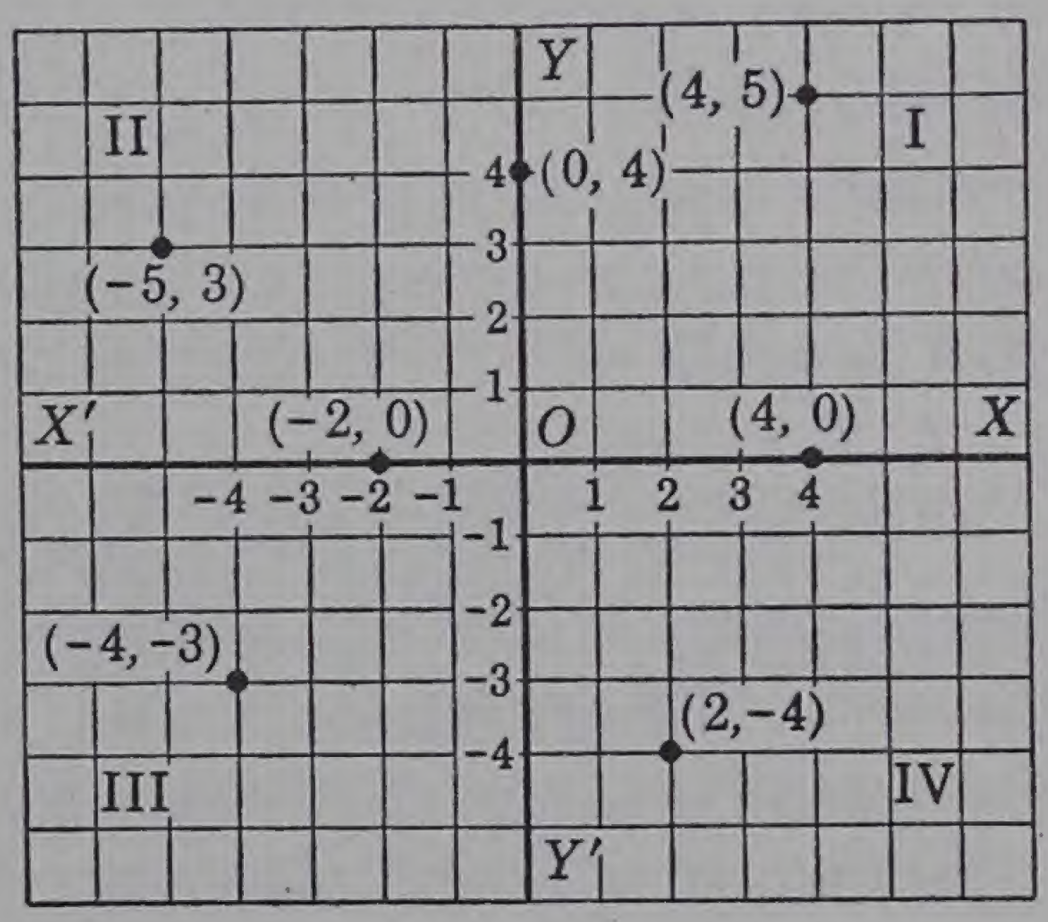


FIG. 6.

the **ordinate.** To determine on which side of the coördinate axes the point lies, we employ the plus and minus signs. The abscissa is positive if measured to the right of the y-axis, and negative if measured to the left of the y-axis, and the ordinate is positive if measured above the x-axis and negative if measured below the x-axis. Thus, the abscissa is positive in quadrants I and IV and negative in quadrants II and III. The ordinate is positive in quadrants I and II and negative in quadrants III and IV. For every point in the plane there is associated but one pair of values; that is, each point has but one abscissa and one ordinate. Conversely, for every pair of values there is but one point in the plane that has the first number for its abscissa and the second number for its ordinate. The abscissa and ordinate of any point are called the **coördinates** of that point.

The pair of values 2 and -4 will locate a point 2 units to the right of the y-axis and 4 units below the x-axis. The symbol $(2, -4)$ is used to represent this point. *The general symbol* (x, y) *denotes the coördinates of a point whose abscissa is* x *and whose ordinate is* y. The process of locating and marking a point whose coördinates are given is called **plotting** the point. Fig. 6 illustrates the plotting of the points $(4, 5)$, $(0, 4)$, $(-5, 3)$, $(-2, 0)$, $(4, 0)$, $(-4, -3)$, and $(2, -4)$.

Plotting is facilitated by the use of coördinate or graph paper. Graph paper is made by ruling off the plane into equal squares with sides parallel to the coördinate axes. With coördinate paper the distances from the coördinate axes are readily determined.

Rectangular coördinates are sometimes called Cartesian coördinates in honor of Descartes, who, in 1637, published a piece of work that to some extent treated geometry analytically and that influenced the development of analytic geometry. The idea of coördinates as applied to the study of geometry seems to have been first used by the Greek Apolonius two centuries before Christ, but not with all its modern meaning. Even at the time of Descartes the y-axis was not used, nor was the idea of positive and negative abscissas and ordinates used. The terms abscissa and ordinate were not used by Descartes. John Wallis, in 1673, was the first to use positive and negative abscissas and ordinates. The French mathematician, G. W. Leibniz (1646–1716), in 1692, was the first to use the terms coördinates and coördinate axes, and was the first to use both abscissa and ordinate. Perhaps the term

ordinate was first used by Kepler, in 1640, and the term abscissa was first used by Cavalieri, in 1653.

EXERCISES

1. Plot the points (2, 3), (−3, 5), (5, −5), (6, 0), (0, −5), (−3, 0), (0, 6), (2, 0).

2. What are the coördinates of the vertices of a square of side 10 with center at the origin and sides parallel to the coördinate axes?

3. What are the coördinates of the vertices of a square of side 8 with center (−2, 3) and sides parallel to the coördinate axes?

4. The point (−2, 1) bisects a line joining two points. If the coördinates of one of these points are (3, −5), what are the coördinates of the other point?

5. A rectangle of sides 5 and 9 has one vertex at (−3, 2). What are the coördinates of the other vertices if the sides are parallel to the coördinate axes and the rectangle lies to the right of the given point? (Consider all cases.)

6. The coördinates of two vertices of an equilateral triangle are (5, 4) and (−3, 4). What are the coördinates of the other vertex?

7. Where are all points whose abscissas are 0? How might this be expressed algebraically?

8. Where are all points whose ordinates are 0? How might this be expressed algebraically?

9. Where are all points whose abscissas are 5? Whose abscissas are −7? How might each of these be expressed algebraically?

10. Where are all points whose ordinates are 3? Whose ordinates are −1? How might each of these be expressed algebraically?

11. Where are all points whose ordinates and abscissas are equal? How might this be expressed algebraically?

12. Where are all points whose ordinates are two units greater than the abscissas? How might this be expressed algebraically?

13. Where are all points whose ordinates are twice the abscissas? How might this be expressed algebraically?

40. GRAPHING ALGEBRAIC FUNCTIONS

In the equation $y = f(x)$, if $f(x)$ is given, there is an unlimited number of pairs of values for x and y that will satisfy this equation. The **graph** of the equation $y = f(x)$ is the **curve** that contains all the points whose coördinates satisfy the equation, but **no other points.** By the **graph of a function** $f(x)$ we mean the graph of the equation $y = f(x)$. Observe that the ordinate of a point with abscissa a on the graph of the function $f(x)$ is $f(a)$; that is, the ordinate is the value of $f(x)$ when a has been substituted for x.

The graph of the function $f(x)$ is approximated by drawing a smooth curve through a set of plotted points. The coördinates of the plotted points are found by assigning values to x and solving for the corresponding values of $f(x)$.

Since only a finite number of points can be found and plotted, it follows that the resulting graph will be an approximate graph; but the more points that are plotted in a given interval, the more accurate will be the graph.

Example 1. Graph the function $x^2 - x - 6$.

Solution. Let y or $f(x)$ represent the value of this function and write

$$y = x^2 - x - 6,$$

or write

$$f(x) = x^2 - x - 6.$$

Let us assign values to x and solve for the corresponding values of y or $f(x)$; and write the results in the accompanying table form, where $f(-5) = 24$, $f(-4) = 14$, $f(-3) = 6$, and so on. These points are plotted and the graph is drawn in Fig. 7.

x	y or $f(x)$
−5	24
−4	14
−3	6
−2	0
−1	−4
0	−6
1	−6
2	−4
3	0
4	6
5	14

Care must be taken to connect the proper points after they have been plotted. It is necessary to connect points arising from successive values of x, *provided the value of the function exists for all values of* x *between two successively chosen values of* x. Thus, the point where $x = 1$ must be connected with the points where x is the next value smaller and the next value larger than 1. In Example 1, the point where $x = 1$ must be connected with the points where $x = 0$ and $x = 2$; that is, in Fig. 7 the point $(1, -6)$ must be connected with the point $(0, -6)$ and with the point $(2, -4)$.

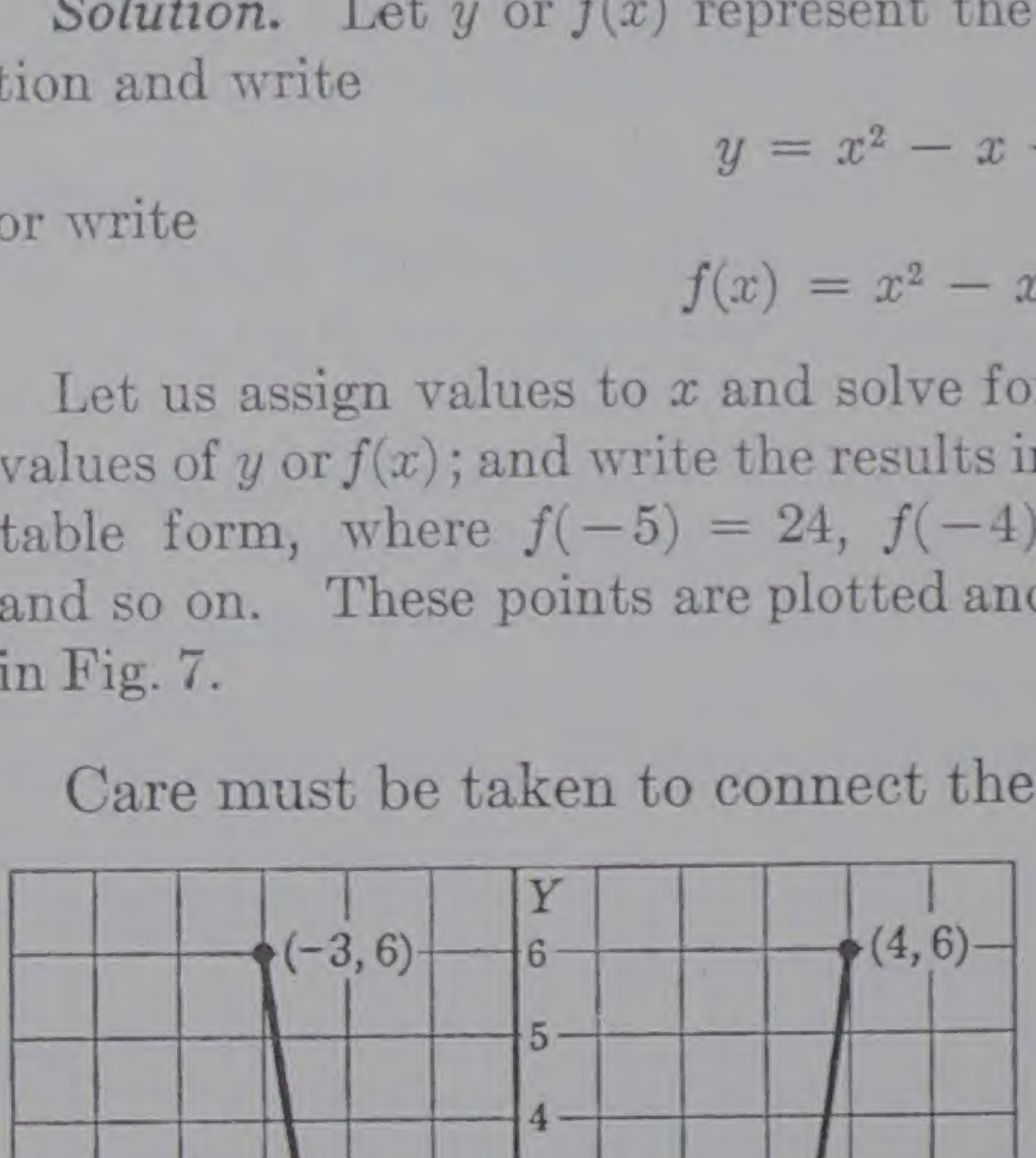


FIG. 7.

The values $x = -2$ and $x = 3$ in Example 1 have a very special significance, because $f(-2) = 0$ and $f(3) = 0$. Thus, the graph crosses the x-axis at the points where $x = -2$ and $x = 3$. The values -2 and 3 are called the zeros of the function $f(x) = x^2 - x - 6$, and the points $(-2, 0)$ and $(3, 0)$ are called the x-intercepts. The point $(0, -6)$ is called the y-intercept.

Those values of x for which $f(x)$ is equal to zero are called the **zeros of the function** $f(x)$. Graphically this means that the **real zeros** of a function $f(x)$ are the abscissas of the points where the graph of $f(x)$ either crosses or touches the x-axis. (See Example 1.) Thus, the real zeros of a function can be found, at least approximately, from the graph of the function.

The abscissas of the points where the graph of the equation $y = f(x)$ crosses or touches the x-axis are called the **x-intercepts,** and the ordinates of the points where the graph crosses or touches the y-axis are called the **y-intercepts.** The x-intercepts are found by setting $y = 0$ in the equation and solving for the corresponding values of x, and the y-intercepts are found by setting $x = 0$ in the equation and solving for the corresponding values of y.

When we graph any function $f(x)$ in a given interval by drawing a smooth curve through a set of points plotted in this interval, we assume that there exists a value for $f(x)$ for every value of x in this interval. This assumption is true for polynomials but is not true for all functions. This prompts the following classification of functions.

A function $f(x)$ is said to be **continuous** for $x = a$ if $f(a)$ is defined and if

$$\lim_{x \to a} f(x) = f(a), \tag{6}$$

when x approaches the limit a in any manner.

A function $f(x)$ is said to be **discontinuous** for $x = a$ if the condition in Equation 6 is not satisfied.

A function $f(x)$ is said to be **continuous in an interval** when it is continuous for all values of x in this interval. In particular, *all polynomials are continuous functions for all values of* x.

A function composed of the ratio of two polynomials, as $\frac{f(x)}{F(x)}$, is continuous for all values of x except those values of x which are zeros of the denominator polynomial $F(x)$.

The following are examples of discontinuous functions: $\frac{x}{x-3}$ is discontinuous at $x = 3$; $\frac{(x-3)(x+1)}{(x-2)(x+4)}$ is discontinuous at $x = 2$ and $x = -4$.

A rational fractional function is discontinuous for those values of x *that make the denominator equal to zero.*

We shall now illustrate the graphing of a discontinuous function.

Example 2. Graph $f(x) = \frac{(x-3)(x+1)}{(x-2)(x+4)}$. (See Fig. 8.)

Solution. If we assign the following values to x and solve for the corresponding values of $f(x)$, we have

x	-5	-3	-1	0	1	3	5
$f(x)$	$\frac{32}{7}$	$-\frac{12}{5}$	0	$\frac{3}{8}$	$\frac{4}{5}$	0	$\frac{4}{9}$

In this case, the point $(1, \frac{4}{5})$ does not connect with the next successively chosen point in the table of data, namely, $(3, 0)$; for, upon examination we see that there does not exist a value for $f(x)$ when $x = 2$, and that $f(x)$ is positive for all values of x between 1 and 2 but is negative for all values of x between 2 and 3. Likewise, the point $(-5, \frac{32}{7})$ cannot be connected to the point $(-3, -\frac{12}{5})$ because there does not exist a value for $f(x)$ when $x = -4$.

To graph expressions that are the product of linear factors, or the product and quotient of linear factors, with no factors common to both numerator and denominator: First, determine those values of x which will make each linear factor equal to zero and mark these values on the x-axis, observing that the graph crosses or touches the x-axis for those values of x which make the numerator zero and that the ordinate of the graph becomes indefinitely large for those values of x which make the denominator zero. Second, determine what sign the function has for values of x between the successive pairs of values of x that have been found; doing this will determine on which side of the x-axis the curve lies in each such interval.

In Example 2, the points of division on the graph are $x = -4$, $x = -1$, $x = 2$, and $x = 3$. These points are shown in Fig. 8. For $x = 3$ and $x = -1$ the function is zero and the graph either crosses or touches the x-axis. For $x = 2$ and $x = -4$ the function becomes indefinitely large. Let us first study the nature of the graph for positive values of x. Starting with $x = 0$ we see that

the y-intercept is $f(0) = \frac{3}{8}$. For values of x between 0 and 2 the factor $(x - 3)$ is negative, $(x + 1)$ is positive, $(x - 2)$ is negative, and $(x + 4)$ is positive. Therefore, $f(x) = \dfrac{(-)(+)}{(-)(+)}$ is positive in the interval $x = 0$ to $x = 2$, and $f(x)$ is infinite at $x = 2$. Thus, in the interval $x = 0$ to $x = 2$, the ordinate of the graph varies from $\frac{3}{8}$ to infinity through positive values.

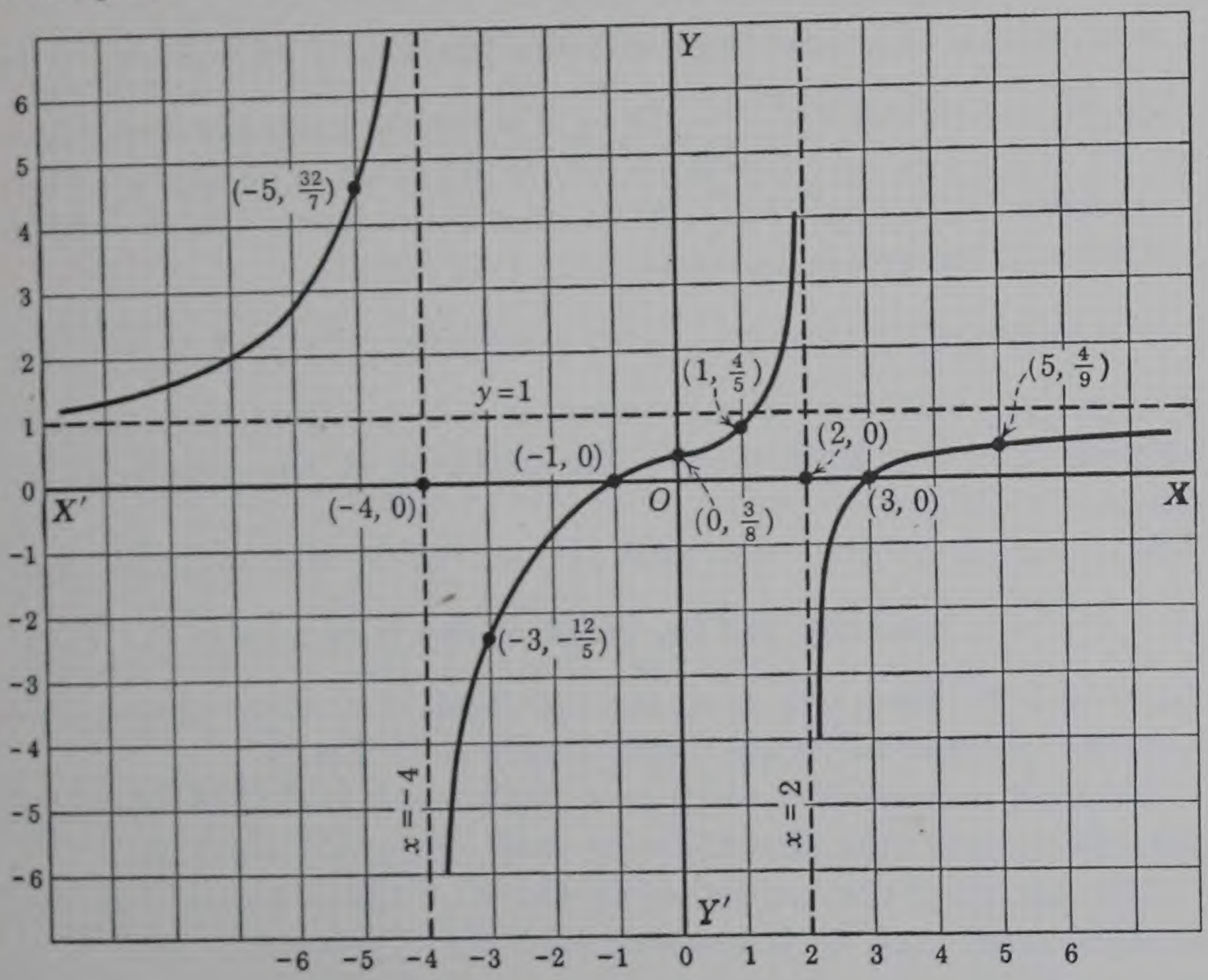


FIG. 8.

For values of x between 2 and 3, we find in a similar manner that $f(x) = \dfrac{(-)(+)}{(+)(+)}$ is negative, and $f(x)$ is zero at $x = 3$. Therefore, in the interval $x = 2$ to $x = 3$, the ordinate of the graph varies from infinity to zero through negative values.

For values of x greater than 3, $f(x) = \dfrac{(+)(+)}{(+)(+)}$ is positive. For very large numerical values of x, it should be observed that the numbers -3, $+1$, -2, and $+4$ in $f(x)$ become insignificant as compared to x, and thus the numerator and denominator of $f(x)$ become approximately equal; that is, $f(x)$ approaches 1 as the

numerical value of x grows larger without bound. Furthermore, since $f(x) = \dfrac{(x-3)(x+1)}{(x-2)(x+4)} = \dfrac{x^2-2x-3}{x^2+2x-8}$, it follows that for very large positive values of x the numerator $x^2 - 2x - 3$ is less than the denominator $x^2 + 2x - 8$, and thus $f(x)$ remains a little less than 1 no matter how large x is. Thus, as x varies from 3 to ∞, the graph increases from zero to 1.

For values of x less than -4 the function $f(x) = \dfrac{(-)(-)}{(-)(-)}$ is positive, the numerator $x^2 - 2x - 3$ is greater than the denominator $x^2 + 2x - 8$, and therefore $f(x)$ is always a little greater than 1 no matter how small x is. Thus, as x varies from $-\infty$ to -4 the ordinate of the graph increases from 1 to infinity.

For values of x between -4 and -1 the function $f(x) = \dfrac{(-)(-)}{(-)(+)}$ is negative, and $f(x)$ is zero when $x = -1$. Therefore, in the interval $x = -4$ to $x = -1$, the ordinate of the graph varies from $-\infty$ to zero.

For values of x in the interval -1 to 0, the function $f(x) = \dfrac{(-)(+)}{(-)(+)}$ is positive, and the graph varies from zero to $\frac{3}{8}$. With these facts in mind and using the coördinates of a few points, it is possible to draw an approximate graph, as in Fig. 8.

We shall now show how to determine the nature of the graph of a fractional function when the numerical values of x are large.

If $f(x)$ is a fraction whose numerator is of **lower** degree than the denominator, then it is evident that $f(x)$ approaches zero as x increases indefinitely or as x decreases indefinitely.

If the numerator of $f(x)$ is of the **same** degree or of **higher** degree than the denominator, the nature of the graph of $f(x)$ for large numerical values of x can best be studied by first dividing the numerator by the denominator and expressing $f(x)$ as:

$$f(x) = Q(x) + \frac{\text{remainder}}{\text{denominator}},$$

where the remainder is of lower degree than the denominator. Since the fraction $\dfrac{\text{remainder}}{\text{denominator}}$ approaches zero as the numerical value x increases indefinitely, it follows that the graph of $y = f(x)$ approaches the graph of the quotient $y = Q(x)$ as x increases

indefinitely. By observing whether the fraction $\frac{\text{remainder}}{\text{denominator}}$ is positive or negative for large numerical values of x, it is possible to determine whether the graph of the given function $f(x)$ lies above or below the graph of $y = Q(x)$ (the quotient).

The limiting position of a tangent to a curve, as the point of contact recedes to infinity along the curve, is called an ***asymptote,*** provided this line passes through a finite portion of the plane. Thus, an asymptote is sometimes spoken of as "a tangent to the curve at infinity." (For the definition of a tangent line, see Article 41.)

In Example 2, $x = 2$ and $x = -4$ are vertical asymptotes, and since

$$f(x) = \frac{(x-3)(x+1)}{(x-2)(x+4)} = \frac{x^2-2x-3}{x^2+2x-8} = 1 - \frac{4x-5}{x^2+2x-8}, \quad (7)$$

it follows that $y = 1$ is a horizontal asymptote. It also follows from Equation (7) that $f(x)$ is less than 1 for large positive values of x, but $f(x)$ is greater than 1 if x is negative and its numerical value is large. That $f(x)$ is greater than 1 if x is less than -4 is apparent, since $\frac{4x-5}{x^2+2x-8}$ is negative if x is less than -4.

The discussion given in this example might be expressed better in symbolic form by making use of the inequality signs $>$ and $<$. The inequality sign $>$ is used to denote greater than, and the inequality sign $<$ is used to denote less than; that is, the open side is toward the greater and the point is toward the smaller. Since we read from left to right, $(a > b)$ is read "***a*** *is greater than* ***b,***" $(a < b)$ is read "***a*** *is less than* ***b,***" and $(a > x > b)$ is read "***a*** *is greater than* ***x*** *and* ***x*** *is greater than* ***b.***" However, $a > b$ also implies that b is less than a.

By making use of the inequality signs we may now analyze the graph (see Fig. 8) of the discontinuous function

$$f(x) = \frac{(x-3)(x+1)}{(x-2)(x+4)},$$

given in Example 2, in the following manner. The zeros of $f(x)$ are $x = -1$ and $x = 3$, and the infinities of $f(x)$ are $x = -4$ and $x = 2$. Arranging these values of x in order of magnitude we have $x = -4, -1, 2$, and 3. Thus:

If	$x < -4$,	then	$f(x) > 1$.
If	$x = -4$,	then	$f(x) = \infty$.
If	$-4 < x < -1$,	then	$f(x) < 0$.
If	$x = -1$,	then	$f(x) = 0$.
If	$-1 < x < 2$,	then	$f(x) > 0$.
If	$x = 2$,	then	$f(x) = \infty$.
If	$2 < x < 3$,	then	$f(x) < 0$.
If	$x = 3$,	then	$f(x) = 0$.
If	$x < 3$,	then	$0 < f(x) < 1$.

We shall next illustrate the graphing of a function in which both numerator and denominator have a common factor.

Example 3. Graph $f(x) = \dfrac{x^2 - 4}{x - 2}$.

Solution. After computing the accompanying table and plotting the corresponding points in Fig. 9, we see that the graph of $f(x) = \dfrac{x^2 - 4}{x - 2}$ is a straight line except for a miniature gap at $x = 2$. Since the value of the function does not exist for $x = 2$, there is no graph at this point. However, $f(x) = \dfrac{x^2 - 4}{x - 2} = x + 2$ for all values of x except $x = 2$. Therefore, the graph of $f(x)$ is the graph of the line $y = x + 2$ except at the point where $x = 2$, unless we assign to $f(x)$ the value 4 when $x = 2$.

x	$f(x)$
−3	−1
−2	0
−1	1
0	2
1	3
1.9	3.9
1.99	3.99
2	?
2.01	4.01
2.1	4.1
3	5

The following example is given to illustrate that caution is necessary when graphing $y = f(x)$ if we plot points obtained by assigning values to x and solving for the corresponding values of y, and then draw a smooth curve through these points, even though $f(x)$ is a continuous function. The resulting graph may not come close to approximating the true graph if the true graph is very irregular between two successive values chosen for x. To illustrate:

Example 4. Graph $y = 4x^3 - 12x^2 + 9x - 1$.

Solution. Let us compute the following table by using integral values for x,

x	−1	0	1	2	3
y	−26	−1	0	1	35

and then plot the corresponding points in Fig. 10. In this case the irregularity in the curve between $x = 0$ and $x = 2$ is liable to be missed.

However, had we computed the following table and plotted the corresponding points, the resulting graph would have been a closer approximation to the true graph.

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	3
y	-26	-9	-1	1	0	-1	1	35

When graphing a given function, it is necessary to choose such intervals for graphing, and to calculate the coördinates of such points, that when a smooth curve is drawn through the plotted points the graph will have at least the following characteristics: (1) *The graph should show all the* x*-intercepts and all the* y*-intercepts.* (2) *The graph should show the asymptotes, if any.* (3) *The graph should show all the turning points.* (4) *The graph should be such that it is impossible to draw a straight line to cut the graph in more points than the degree of the function.* The graphs of most functions of small degree are such that it is possible to draw a straight line to cut the graph in exactly as many points as the degree of the given expression. This straight line, however, need not be the x-axis.

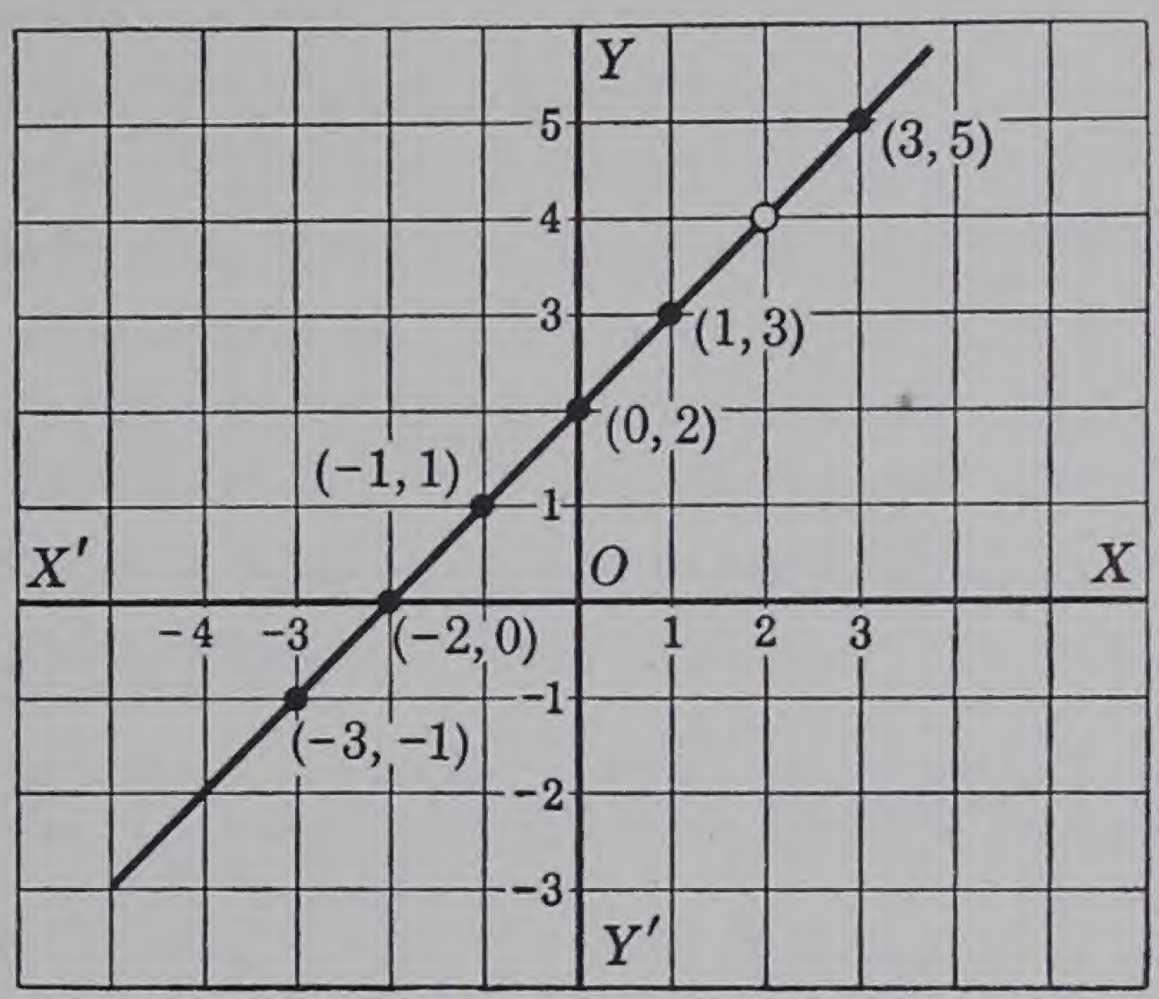


FIG. 9.

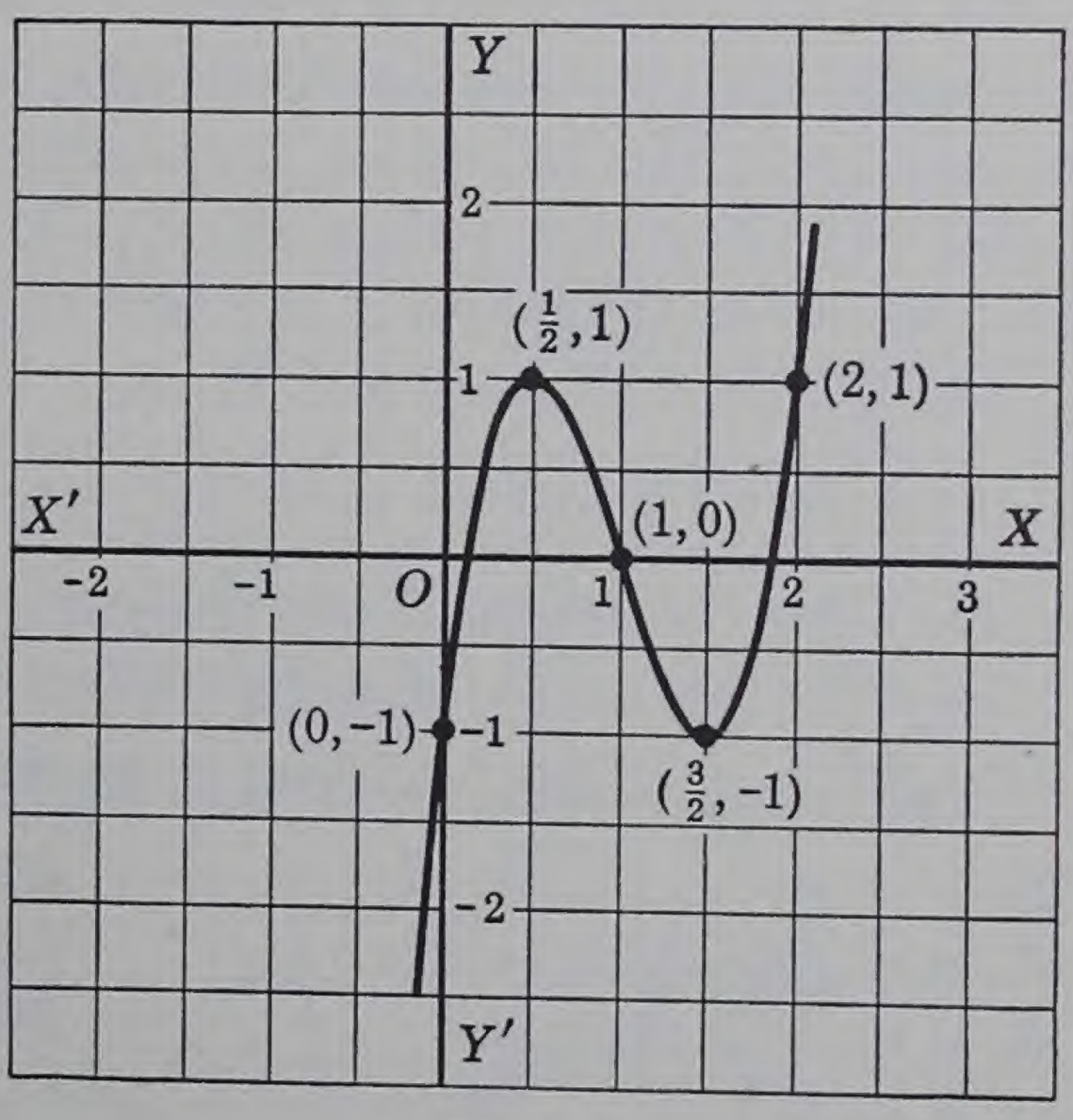


FIG. 10.

By the **degree of a rational fractional expression** is meant the degree (See Article 25) of the expression when cleared of fractions. Thus, $y = \dfrac{1}{x-1}$ is of the second degree because $xy - y - 1 = 0$ is of the second degree in x and y.

QUESTIONS

1. Define coördinate axes.
2. What is meant by the x-axis? By the y-axis?
3. Define abscissa.
4. Define ordinate.
5. What is meant by the coördinates of a point?
6. Define polynomial.
7. What is meant by a rational integral function?
8. What is meant by a rational fractional function?
9. What is meant by the degree of a rational integral function?
10. What is meant by the degree of a rational fractional function?
11. What is meant by the graph of a function of two variables?
12. What is meant by the zeros of a function?
13. What is meant by the x-intercept of a graph? By the y-intercept?
14. How are the intercepts of a graph found?

EXERCISES

Graph each of the following nine equations:

1. $y = x$.
2. $y = 3x$.
3. $y = x + 2$.
4. $y = 3x + 2$.
5. $y = 3x - 2$.
6. $y = 5$.
7. $x - 3y + 4 = 0$.
8. $3x - 2y - 1 = 0$.
9. $2x + 5y + 6 = 0$.

10. In the expression $y = ax + b$, how will an increase or decrease in the value of b, the constant term, affect the graph? How will an increase or decrease in the value of a, the coefficient of x, affect the graph?

11. Which of the points $(2, 4)$, $(3, -4)$, $(3, 4)$, $(1, -3)$, $(-1, 0)$, $(0, -5)$ lie on the graph of $y = 2x^2 - 3x - 5$?

12. Which of the points $(0, \frac{9}{4})$, $(1, -8)$, $(3, 0)$, $(0, -\frac{9}{4})$, $(-3, 0)$ $(2, -5)$, $(-1, \frac{8}{9})$ lie on the graph of $y = \dfrac{x^2 - 9}{(x - 2)^2}$?

13. Give the degree of each exercise from 24 to 29.
14. Give the degree of each exercise from 30 to 38.

Graph each of the functions in Exercises 15 through 38.

15. $y = x^2$.
16. $y = x^2 - 2x$.
17. $y = (x + 1)(x - 2)$.
18. $y = 3x^2$.
19. $y = x^2 - 2x - 3$.
20. $y = (x + 1)(x - 2)(x + 3)$.
21. $y = x^2 + 3$.
22. $y = 3 + 2x - x^2$.
23. $y = (x + 1)(x - 2)^2$.
24. $y = (x - 2)^2$.
25. $y = x^2 + 2x - 3$.
26. $y = (x + 1)^2(x - 2)^2$.
27. $y = \dfrac{x + 5}{x - 2}$.
28. $y = \dfrac{(x - 3)^2}{x + 5}$.
29. $y = \dfrac{(x - 1)(x - 3)}{(x + 2)}$.
30. $y = \dfrac{x^2 - 9}{x - 3}$.
31. $y = \dfrac{(x + 2)^2}{(x - 2)^2}$.
32. $y = \dfrac{(x - 1)}{(x + 2)(x - 3)}$.

33. $y = \dfrac{2x - 3}{x + 2}$.

34. $y = \dfrac{6x^2 + x - 2}{4x^2 + 4x - 15}$.

35. $y = \dfrac{(x - 2)^2}{x(x + 1)(x - 3)}$.

36. $y = \dfrac{x + 5}{(x - 3)^2}$.

37. $y = \dfrac{(x - 2)(x + 3)}{(x - 3)(x + 4)}$.

38. $y = \dfrac{x(x + 3)^2}{(x - 2)(x - 1)}$.

39. Compare the graphs of $y = f(x)$, $y = -f(x)$, and $y = f(-x)$. (See Exercises 19, 22, and 25.)

40. Compare the graphs of $y = f(x)$ and $x = f(y)$. Graph $x = f(y)$ for Exercises 27, 28, 29, 31, 33, and 36; that is, graph $x = \dfrac{y + 5}{y - 2}$, and so on.

41. SLOPE OF A CURVE

A very important aid to graphing is to know in what direction the graph passes through a given point. This direction is best expressed in terms of **slope.**

The slope **m** *of a straight line passing through two points whose coördinates are* (x_1, y_1) *and* (x_2, y_2) *is defined to be*

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \qquad (x_1 \neq x_2) \qquad (8)$$

The slope of a given curve at a given point is defined to be the slope of the tangent line at this point.

From the definition of slope of a straight line, we see that the slope is the same at every point on the line. Thus, if a point moves along a straight line from any point on the line to another point on the line, the ratio of the change in the ordinate of the point to the corresponding change in its abscissa is the slope of the line.

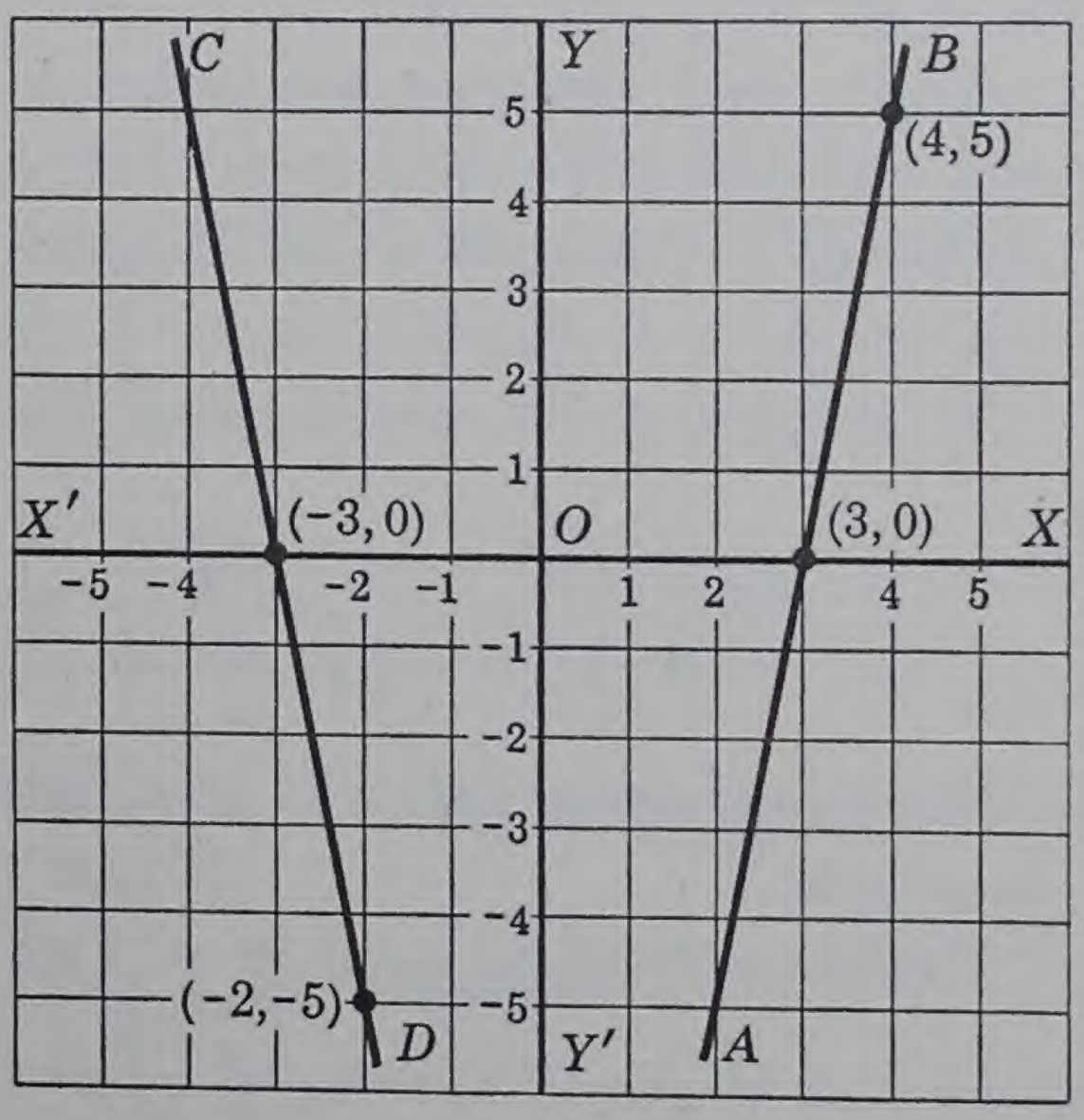


FIG. 11.

A line with slope 5 is such that if a point moves from left to right along the line, the ordinate of the point will increase 5 units for each unit increase in the abscissa; see line AB

in Fig. 11. A line with a slope -5 is a line that falls 5 units for each unit increase in the abscissa; see line CD in Fig. 11.

Since the slope of a curve at a given point is defined to be the slope of the tangent line at this point, we shall now show what is meant by a **tangent line to a curve** at a given point $P(x, y)$ on the curve. Suppose the graph of $y = f(x)$ to be a smooth continuous curve in the neighborhood of the point P on the curve, and suppose a secant line be drawn through P cutting the curve at a second point Q as in Fig. 12. Then, as the point Q moves along the curve and approaches P as a limit, *the secant line in its limiting position is defined to be the tangent line to the curve at the point* $\boldsymbol{P}$, *and the limiting value of the slope of the secant line is defined to be the slope of the curve, at the point* $\boldsymbol{P}$.

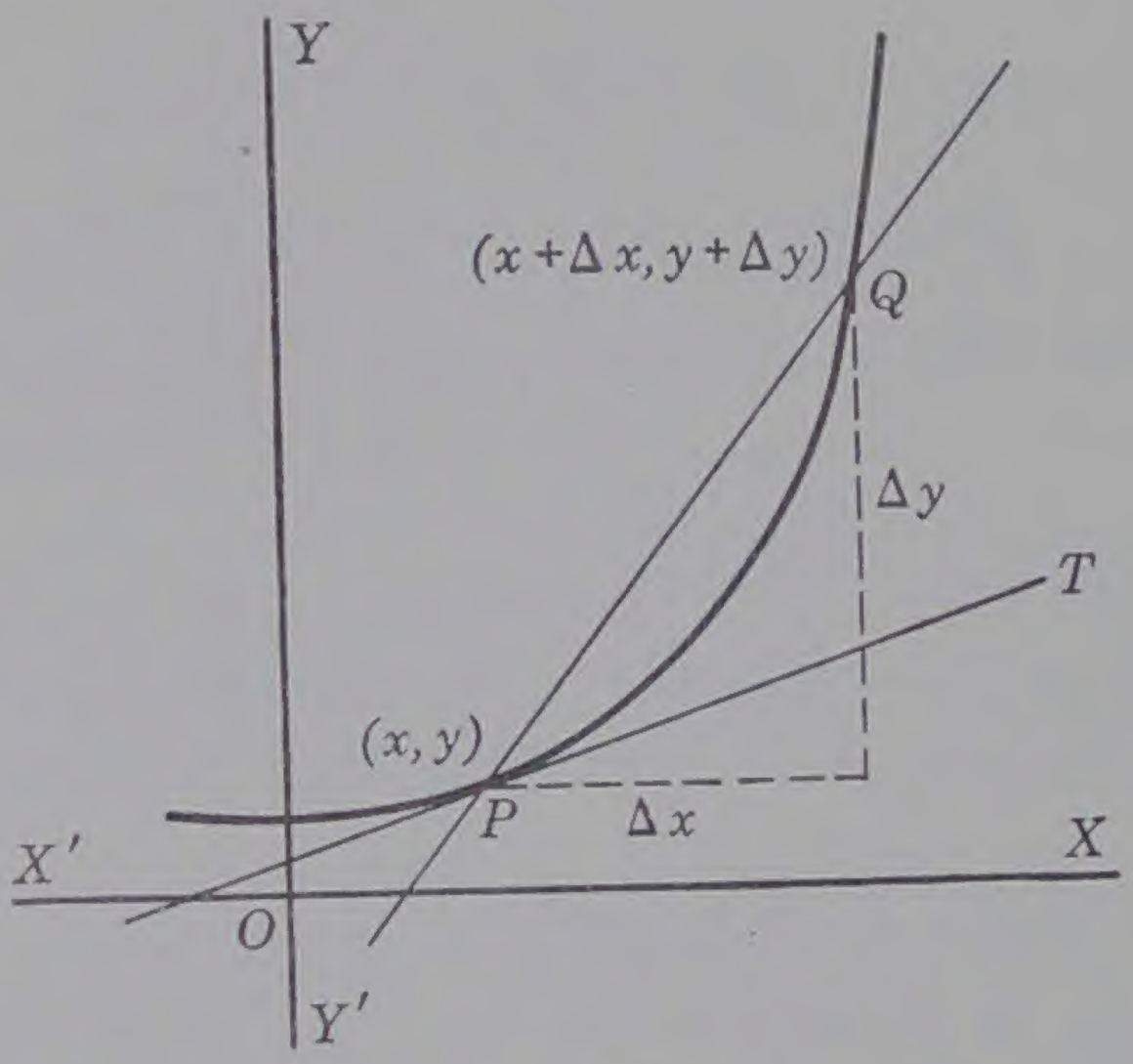


FIG. 12.

It is customary in calculus to represent a small increase in x by the symbol Δx, and a small increase in y by the symbol Δy. This increase from one value to another is called the **increment** of the variable, and we read the symbol Δx as "*the increment of* x" or "*delta* x"; likewise the symbol Δy is read "*the increment of* y" or "*delta* y." Thus, the coördinates of the point Q can now be written as $(x + \Delta x, y + \Delta y)$.

By Equation (8), the slope of the secant line through P and Q is

$$\frac{(y + \Delta y) - y}{(x + \Delta x) - x} = \frac{\Delta y}{\Delta x} = \text{slope of secant line } PQ.$$

Now as Q approaches P as a limit, Δx must approach zero, and therefore:

The slope $\boldsymbol{m}$ *of the curve* $\boldsymbol{y = f(x)}$ *at the point* $\boldsymbol{P(x, y)}$ *on the curve is equal to the limiting value of* $\dfrac{\Delta y}{\Delta x}$ *as* Δx *approaches zero.*

The limiting value of $\dfrac{\Delta y}{\Delta x}$ is called *the derivative of* $\boldsymbol{y}$ *with respect*

to x and is denoted by the symbol $\frac{dy}{dx}$. The symbol $f'(x)$ is also used to denote the derivative. Thus,

$$m = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x) = \textbf{slope of tangent at } \boldsymbol{P(x, y)}. \quad (9)$$

The process of finding the derivative is called **differentiation.**

The value of the derivative at any point on the curve is equal to the slope of the curve at this point.

Since $f'(x_1)$ is the value of the derivative of the function $f(x)$ when $x = x_1$, $f'(x_1)$ is the value of the slope of the tangent line to the graph of $f(x)$ at the point or points on the graph where $x = x_1$, and $f'(x_1)$ is also the value of the slope of the curve of $f(x)$ at the point or points on the curve where $x = x_1$.

We shall now illustrate how the derivative can be found by using the limiting process.

Example 1. Find the slope of the curve of the equation $y = 3x^2$ at the point where $x = 4$.

Solution. Finding this slope is equivalent to finding the value of the derivative of y with respect to x at the point (4, 48). To find $\frac{dy}{dx}$: first, find the value of $\frac{\Delta y}{\Delta x}$; second, find the limiting value of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero.

Let us first consider the general point $P(x, y)$ on the curve, without restricting ourselves to the fixed point (4, 48). Since the coördinates of the points $P(x, y)$ and $Q(x + \Delta x, y + \Delta y)$ must satisfy the given equation $y = 3x^2$, we have for the point P,

$$y = 3x^2,$$

and for the point Q,

$$y + \Delta y = 3(x + \Delta x)^2 = 3[x^2 + 2x\Delta x + (\Delta x)^2].$$

Solving for Δy by subtracting the members of the first equation from the members of the second equation, we have

$$\Delta y = 3(x + \Delta x)^2 - 3x^2 = 6x\Delta x + 3(\Delta x)^2,$$

dividing by Δx gives

$$\frac{\Delta y}{\Delta x} = 6x + 3\Delta x.$$

Taking the limit of both members of this equation as Δx approaches zero, we have

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (6x + 3\Delta x) = 6x,$$

since $3\Delta x$ approaches zero as Δx approaches zero. Thus, the value of the derivative of the function $y = 3x^2$, or the value of the slope of the graph of $y = 3x^2$, at any point $P(x, y)$, is $6x$. The value of the derivative, or the value of the slope to the curve at the point where $x = 4$ is therefore $f'(4) = 6 \cdot 4 = 24$. Hence the slope of the curve of the equation $y = 3x^2$ at the point where $x = 4$ is 24.

The derivative can always be found in the manner given in Example 1. However, we shall derive a general formula for finding the derivative of any power of x, and in the future we shall use this formula when finding the derivative of any polynomial. To derive this formula, consider the following function

$$y = ax^n. \tag{10}$$

Then

$$y + \Delta y = a(x + \Delta x)^n. \tag{11}$$

Subtracting (10) from (11) we have

$$\Delta y = a(x + \Delta x)^n - ax^n; \tag{12}$$

dividing by Δx gives

$$\frac{\Delta y}{\Delta x} = a\,\frac{(x + \Delta x)^n - x^n}{\Delta x}. \tag{13}$$

Since $(x + \Delta x)^n = x^n + nx^{n-1}\Delta x$ plus terms containing $(\Delta x)^2$, $(\Delta x)^3$, and so on (Article 92), it follows that

$$\frac{(x + \Delta x)^n - x^n}{\Delta x} = nx^{n-1} + \text{terms containing } (\Delta x), (\Delta x)^2, \cdots \tag{14}$$

Hence,

$$\lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} = nx^{n-1}, \tag{15}$$

since all the terms containing Δx will approach zero. Therefore,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} a\,\frac{(x + \Delta x)^n - x^n}{\Delta x} = a \cdot nx^{n-1}. \tag{16}$$

The derivative of $\boldsymbol{ax^n}$ *is the product of the coefficient* $\boldsymbol{a}$, *the exponent* $\boldsymbol{n}$, *and* $\boldsymbol{x}$ *with the exponent decreased by one.*

To find the derivative of a polynomial, apply this rule to each term of the polynomial. To illustrate, if

$$y = 5x^4 - 3x^2 + 10x,$$

then

$$\frac{dy}{dx} = 5(4x^3) - 3(2x) + 10 = 20x^3 - 6x + 10.$$

If

$$y = 8x^7 - 3x^5 + 4x^3 - 5x^2 + 12,$$

then

$$\frac{dy}{dx} = 56x^6 - 15x^4 + 12x^2 - 10x.$$

In this article we shall limit our discussion to polynomial functions. The derivative of the general polynomial

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n$$

is

$$\frac{dy}{dx} = a_0nx^{n-1} + a_1(n-1)x^{n-2} + a_2(n-2)x^{n-3} + \cdots + 2a_{n-2}x + a_{n-1}. \qquad (17)$$

Since the derivative of y with respect to x is the rate of change of y with respect to x, we should observe the following:

If $\frac{dy}{dx}$ is positive for all values of x in a given interval, then the value of the function $y = f(x)$ increases in this interval as x increases; if $\frac{dy}{dx}$ is negative for all values of x in a given interval, then the value of the function $y = f(x)$ decreases in this interval as x increases; if $\frac{dy}{dx} = 0$ for any value of x, say $x = a$, then the slope of the graph of $f(x)$ is zero at the point on the curve where $x = a$. Geometrically this means a point moving on the graph of the equation $y = f(x)$ rises as we proceed along the curve from left to right in any interval in which $\frac{dy}{dx}$ is positive, a point moving on the graph of the equation $y = f(x)$ falls as we proceed along the curve from left to right in any interval in which $\frac{dy}{dx}$ is negative, and the graph of the equation $y = f(x)$ has a horizontal tangent at any point on the curve where $\frac{dy}{dx} = 0$.

The points on the graph where the tangent lines are horizontal are called **critical points.** The critical points are given special names. If a graph has a horizontal tangent at a point on the curve where $x = a$, the point is called a **maximum point** if the graph lies entirely below the tangent line in the neighborhood of $x = a$, as at point P_1 in Fig. 13; or the point is called a **minimum point** if the graph lies entirely above the tangent line in the neighborhood of $x = a$, as at point P_3.

If the graph crosses the tangent line (any tangent) at the point of tangency, this point is called a **point of inflection,** as points P_2 and P_4.

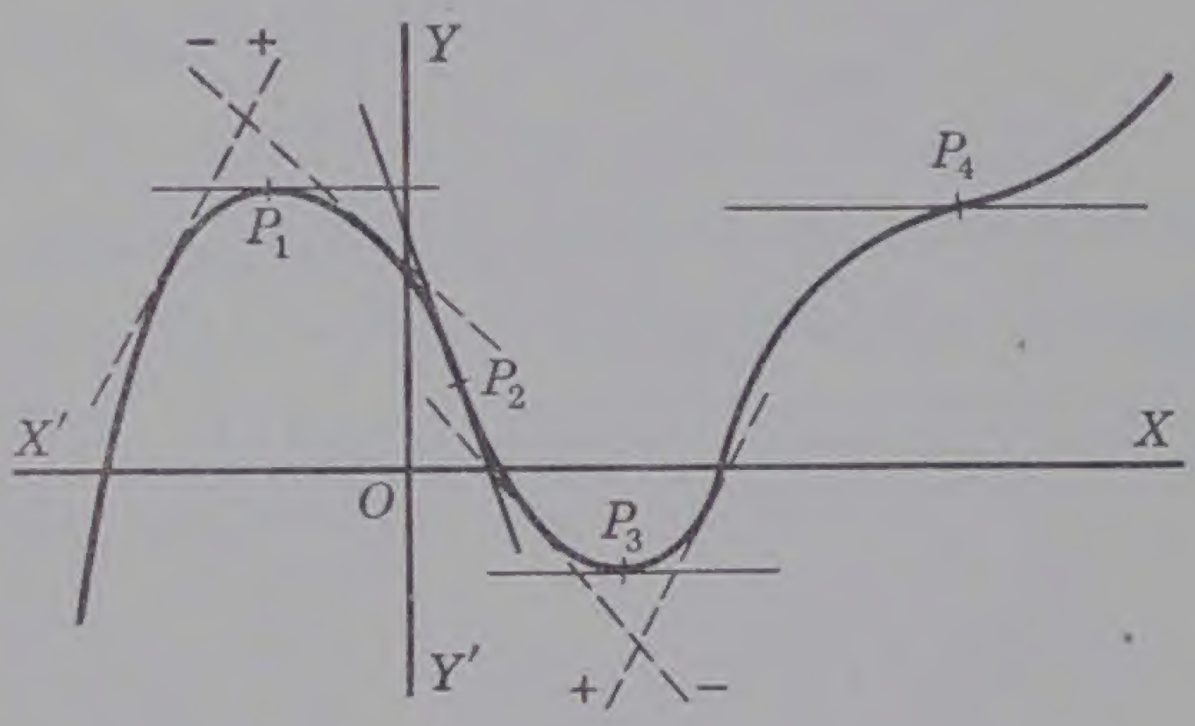


FIG. 13.

It should be observed that the slope of the curve changes from positive values to zero to negative values as a point passes from left to right along the graph through a maximum point such as P_1, and that the slope of the curve changes from negative values to zero to positive values as a point passes from left to right along the curve through a minimum point such as P_3 in Fig. 13.

When the curve $f(x)$ has a maximum point at $x = a$, the function $f(x)$ is said to have a **maximum value** for $x = a$, and the maximum value is $f(a)$.

When the curve $f(x)$ has a minimum point at $x = a$, the function $f(x)$ is said to have a **minimum value** for $x = a$, and the minimum value is $f(a)$.

Since the slope of the graph of the function $f(x)$ is zero at the critical points, the critical values of x of any function $f(x)$ are found by solving for the real roots of the equation

$$f'(x) = 0. \tag{18}$$

This equation is formed by setting the derivative of $f(x)$ equal to zero.

To determine whether the function $f(x)$ is a maximum or a minimum, or neither, for the critical value $x = a$, we apply the following test:

The function $f(x)$ *is a maximum for* $x = a$ *if* $f'(x)$ *is positive for values of* x *slightly less than* a, *is zero for* $x = a$, *and is negative for*

values of x *slightly greater than* a. *The maximum value of* $f(x)$ *is* $f(a)$.

The function $f(x)$ *is a minimum for* $x = a$ *if* $f'(x)$ *is negative for values of* x *slightly less than* a, *is zero for* $x = a$, *and is positive for values of* x *slightly greater than* a. *The minimum value of* $f(x)$ *is* $f(a)$.

Example 2. Find the maximum and minimum points of the curve

$$y = f(x) = 4x^3 - 12x^2 + 9x - 1.$$

Solution. Differentiating $y = 4x^3 - 12x^2 + 9x - 1$ and factoring, we have

$$\begin{aligned}\frac{dy}{dx} = f'(x) &= 12x^2 - 24x + 9 \\ &= 3(4x^2 - 8x + 3) \\ &= 3(2x - 1)(2x - 3).\end{aligned}$$

Setting $\frac{dy}{dx}$ equal to zero gives

$$3(2x - 1)(2x - 3) = 0;$$

then

$$2x - 1 = 0 \quad \text{or} \quad 2x - 3 = 0.$$

Solving each equation for x, we obtain

$$x = \tfrac{1}{2} \quad \text{or} \quad x = \tfrac{3}{2}.$$

Hence, the critical values of x in the function $4x^3 - 12x^2 + 9x - 1$ are $x = \frac{1}{2}$ and $x = \frac{3}{2}$, and the graph of this function (see Fig. 10) has a horizontal tangent at the points on the curve where $x = \frac{1}{2}$ and $x = \frac{3}{2}$.

Applying the test for maximum and minimum, we find that, for x a little less than $\frac{1}{2}$, the slope $f'(x) = 3(2x - 1)(2x - 3)$ is positive because both $(2x - 1)$ and $(2x - 3)$ are negative; for $x = \frac{1}{2}$ the slope is zero; and for x a little greater than $\frac{1}{2}$, the slope $f'(x) = 3(2x - 1)(2x - 3)$ is negative because $(2x - 1)$ is positive and $(2x - 3)$ is negative. Hence, the function is a maximum for $x = \frac{1}{2}$, the maximum value of the function is $f(\frac{1}{2}) = 4(\frac{1}{2})^3 - 12(\frac{1}{2})^2 + 9(\frac{1}{2}) - 1 = 1$, and the maximum point is $(\frac{1}{2}, 1)$.

Testing $x = \frac{3}{2}$ in the same manner, we find that $f'(x)$ changes from negative values to zero to positive values as x passes from left to right along the curve through $x = \frac{3}{2}$. Hence, the function is a minimum for $x = \frac{3}{2}$, the minimum value of the function is $f(\frac{3}{2}) = 4(\frac{3}{2})^3 - 12(\frac{3}{2})^2 + 9(\frac{3}{2}) - 1 = -1$, and the minimum point is $(\frac{3}{2}, -1)$.

Let us illustrate how the derivative may be used to graph the function in this example. We have just found the maximum and

the minimum point on the graph, and a complete analysis of the graph may be given as follows:

$$f(x) = y = 4x^3 - 12x^2 + 9x - 1$$
$$f'(x) = \frac{dy}{dx} = 12x^2 - 24x + 9$$
$$= 3(2x - 1)(2x - 3).$$

If $x < \frac{1}{2}$, then $\frac{dy}{dx} > 0$, since the signs of $f'(x)$ are $(-)(-)$.

If $x = \frac{1}{2}$, then $\frac{dy}{dx} = 0$.

If $\frac{1}{2} < x < \frac{3}{2}$, then $\frac{dy}{dx} < 0$ since the signs of $f'(x)$ are $(+)(-)$.

If $x = \frac{3}{2}$, then $\frac{dy}{dx} = 0$.

If $x > \frac{3}{2}$, then $\frac{dy}{dx} > 0$ since the signs of $f'(x)$ are $(+)(+)$.

Thus we see that the graph of $y = f(x) = 4x^3 - 12x^2 + 9x - 1$ is rising for $x < \frac{1}{2}$, has a maximum point at $\left(\frac{1}{2}, 1\right)$, is falling for $\frac{1}{2} < x < \frac{3}{2}$, has a minimum point at $\left(\frac{3}{2}, -1\right)$, and is rising for $x > \frac{3}{2}$. (See Fig. 10).

EXERCISES

Find $\frac{dy}{dx}$ for each of the functions in Exercises 1 through 18.

1. $y = 3x$.
2. $y = 3x + 5$.
3. $y = 5 - 3x$.
4. $y = 3x^4 - 5x^2 + 2x - 1$.
5. $y = x^{-4} + 4x^{-2} + 7x - 2$.
6. $y = x^{\frac{2}{3}} - 3x^{\frac{3}{4}} + 2x - 3$.
7. $y = x^2$.
8. $y = x^2 - 5x$.
9. $y = x^2 - 5x + 4$.
10. $y = 2x^3 - 5x^2 + 7x - 2$.
11. $y = 3x^{-5} + 2x^{-3} - 4x^{-2} - 6$.
12. $y = 3x^5 - 3x^4 + 2x^3 - 5x - 6$.
13. $y = x^2 - 2x - 3$.
14. $y = 3 + 2x - x^2$.
15. $y = x^2 + 2x - 3$.
16. $y = \frac{3}{4}x^4 - \frac{2}{3}x^2 - 3x + 7$.
17. $y = 2x^{-4} - 3x^3 + 2x^{\frac{1}{2}}$.
18. $y = 8\sqrt[4]{x} - 3\sqrt[3]{x} + 2\sqrt{x} - 5x$.

Find the value of the slope for each of the following curves (Exercises 19 through 26) at the point where x has the indicated value.

19. $y = 2x^2$	$(x = 0)$.
20. $y = 3x^2 - x + 5$	$(x = -2)$.
21. $y = 3x^2 + 1$	$(x = 1)$.
22. $y = x^2 + x + 1$	$(x = -3)$.
23. $y = 2 - x^2$	$(x = 2)$.
24. $y = 5x$	$(x = 5)$.
25. $y = 2x^3 - 3x^2 + 4x - 5$	$(x = -1)$.
26. $y = 3x^3 - 4x^2 - 2x + 4$	$(x = 3)$.

Find the maximum and minimum points on each of the following curves (Exercises 27 through 35).

27. $y = x^2 - 6x - 7$.
28. $y = 4x^2 - 6x + 5$.
29. $y = 5 - 8x - 4x^2$.
30. $y = -16 + 12x - 3x^2$.
31. $y = 2x^3 - 6x - 3$.
32. $y = 2x^3 - 9x^2 + 12x - 3$.
33. $y = 6 + 3x^2 - x^3$.
34. $y = x^3 + 6x^2 + 12x + 5$.
35. $y = 8 - 9x^2 - 3x^3$.

36. Find the slope, then draw the tangent lines to the curve $y = x^3 - 3x^2 - 9x + 12$ through the following points: $(-3, -15)$, $(-2, 10)$, $(-1, 17)$, $(0, 12)$, $(1, 1)$, $(2, -10)$, $(3, -15)$, $(4, -8)$, and $(5, 17)$. Then draw the curve of $y = x^3 - 3x^2 - 9x + 12$.

37. Show that the slope of the curve $y = x^3 + 5x + 7$ is always positive.

38. The work W done by exploding gas with x cubic feet of air is $W = 80x - 3.2x^2$. Find the mixture that will give the maximum amount of work.

39. Find two numbers whose sum is 20 and whose product is a maximum.

40. What is the largest rectangular area that can be enclosed with 100 feet of fence?

41. What is the largest rectangular area that can be enclosed with 100 feet of fence if one side of the field is along a stone wall and doesn't need a fence?

42. The distance S in miles a train travels from a given station in t hours is given by $S = 20t - t^2$. Find the velocity of the train when $t = 0$; $t = 5$; $t = 10$; $t = 15$. What is the maximum distance the train traveled from the station?

43. The height S in feet reached by a stone thrown vertically into the air with a velocity of V_0 feet per second is given by $S = V_0t - \frac{1}{2}gt^2$. (a) Find the velocity at any time T. (b) Find the maximum height the stone will reach.

44. Graph each of the following equations in the indicated intervals by calculating the ordinate and slope at each point for the integral values of x in the indicated intervals:

(a) $y = x^3 - 3x^2 + 4$ from $x = -2$ to $x = 3$.

(b) $y = x^3 - x^2 - x - 4$ from $x = -2$ to $x = 3$. (Include the point where $x = -\frac{1}{3}$.)

(c) $y = x^3 - 3x + 6$ from $x = -2$ to $x = 3$.

(d) $y = 2x^3 - 5x^2 - 9x + 18$ from $x = -3$ to $x = 4$.

(e) $y = 2x^3 - 3x^2 - x + 1$ from $x = -3$ to $x = 4$.

CHAPTER VII

Linear Equations in One Unknown

42. DEFINITION OF AN EQUATION

An *equation* is a statement that two algebraic expressions are equal. The two expressions are called the **members** of the equation. The statement that two polynomials are equal is called an **algebraic equation.** It is always possible to write any algebraic equation in the following form, where $f(x)$ is a polynomial. This form we shall call the standard form.

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n = 0, \qquad (a_0 \neq 0) \qquad (1)$$

where n is the degree of the equation and must be a **positive integer,** and the a's, the coefficients, are any constants, real or imaginary. Observe that if n is a positive integer, then all the remaining exponents of x in any algebraic equation must be positive integers. Equations of the first, second, third, fourth, fifth, sixth, are called *linear, quadratic, cubic, quartic, quintic, sextic equations,* respectively.

Equalities of the type

$$(x - 2)^2 = x^2 - 4x + 4 \qquad (2)$$

are usually spoken of as **identities** instead of equations because they are satisfied for all values of the letters for which the members are defined. Thus, identities have an infinite number of solutions.

In a function or polynomial in x, the x can usually assume any value whatsoever. However, in an equation in x, the x can have only those particular values which will make both members of the equation equal. This is why, in an equation of one letter, we speak of this letter as being the **unknown** instead of a variable. An equation whose members are equal for a finite number of values of the unknown is called a **conditional equation.** To illustrate, in

the equation

$$2x - 5 = 3,$$

$2x - 5$ will equal 3 only on the condition that $x = 4$.

All conditional equations in this course will be called equations.

43. ROOTS OF AN EQUATION

To solve an equation in one unknown is to find all the values of the unknown, called **roots** *or* **solutions** *of the equation, which are such that when each value is substituted for the unknown in the given equation it will satisfy the equation.* That such values for the unknown exist for algebraic equations follows from **the fundamental theorem of algebra,** which states that: *Every algebraic equation has as least one root, real or imaginary.* This theorem will be assumed without proof, as the proof is beyond the scope of this book.*

As a result of the fundamental theorem of algebra it can be shown that every algebraic equation of degree n must have exactly n roots. See Article 68, Chapter XII. (The n roots need not all be distinct.)

To solve for the roots of the equation $f(x) = 0$ is equivalent to finding the zeros of the function $f(x)$, for the zeros of the function $f(x)$ are the roots of the equation $f(x) = 0$. Thus, the real roots of the equation $f(x) = 0$ are the x-intercepts of the graph of $y = f(x)$.

Graphical Solution. *The real roots of the equation* $f(x) = 0$ *are the* x*-intercepts, or the abscissas, of the points where the graph of* $y = f(x)$ *crosses or touches the* x*-axis.*

44. OPERATING ON EQUATIONS

In the process of solving equations in one unknown, it is usually necessary to rewrite the given equation in several different forms. This is accomplished by either adding, subtracting, multiplying, or dividing both members of the given equation, or of any of the derived equations, by the same expression; or by taking the same power or root of both members of the given equation, or of any of the derived equations. When solving equations it is necessary to know whether all the roots of the given equation are included in the roots of the final equation and whether all the roots of the final equation are roots of the given equation.

* For a proof, see Dickson's *Elementary Theory of Equations*, pp. 47–53.

Two equations are said to be **equivalent** if each equation is satisfied by all the roots of the other equation.

If a derived equation contains all the roots of the original equation and some others, we say the derived equation is **redundant** with respect to its original equation, and the extra roots are called **redundant** or **extraneous** roots.

If the derived equation lacks some of the roots of the original equation, we say the derived equation is **defective** with respect to its original equation.

It is, therefore, helpful to know what operations always lead to equivalent equations, and what operations may lead to defective or redundant equations.

We can add, subtract, multiply, or divide both members of an equation by the same **constant** *(multiplication and division by zero excepted), or we can add to or subtract from both members of an equation the same function of the unknown and the resulting equations will always be equivalent to the original equation. If we extract the same root, say the* **n***th root, of both members of an equation, we get* **n** *different equations and the solutions of these* **n** *equations are the same as the* **n** *solutions of the original equation.*

That is, such operations as transposing, or reducing the equation by dividing each member of the equation by a common constant factor, or dividing each member of the equation by the constant coefficient of the unknown, or extracting the square root of both members of the equation, provided we keep both the positive and the negative root, always lead to equivalent equations.

However, there are three operations that may lead to equations that are not equivalent to the original equation. These operations are: (1) *Raising both members of an equation to the same power, which may lead to a redundant equation;* (2) *multiplying both members of an equation by the same function of the unknown, which may lead to a redundant equation;* (3) *dividing both members of an equation by the same function of the unknown, which may lead to a defective equation.*

Whenever the first two of these operations are employed in the process of solving a given equation, it is necessary to check each root of the resulting equation by substituting it in the given equation to see whether it is a root of the given equation or whether it is a redundant root. Operation (3) should be used with caution, because this operation leads to defective equations, and a root once lost is difficult to find.

We shall show by the following examples how these three operations may lead to redundant or defective equations.

Example 1. In this example we illustrate what may happen when the same power of both members of an equation is taken.

Suppose we start with the linear equation

$$x = 2,$$

and square both members, thus:

$$x^2 = 4;$$

then, solving, we get

$$x = \pm \sqrt{4};$$

hence

$$x = +2 \quad \text{or} \quad x = -2.$$

Since the given equation is linear it has only one root. The second equation is a quadratic and hence has two roots. Therefore, the two equations are not equivalent. The second equation is redundant and has the extraneous root -2.

Example 2. This example is another illustration of taking the same power of both members of an equation.

Suppose we wish to solve the equation

$$\sqrt{x} + 1 = 0.$$

Transposing, we have

$$\sqrt{x} = -1;$$

then, squaring both members gives

$$x = 1.$$

However, 1 does not satisfy the original equation $\sqrt{x} + 1 = 0$ because $\sqrt{1} + 1 = 2$ and not 0. Hence 1 is not a root of the given equation. Thus, after squaring both members of the equation $\sqrt{x} = -1$, the derived equation is not equivalent to the original equation.

If the same power of both members of an equation is taken, the derived equation may have roots that are not roots of the original equation.

Example 3. Solve the equation

$$\sqrt{x} - 1 = 0.$$

Transposing, we have

$$\sqrt{x} = 1;$$

squaring both members gives

$$x = 1.$$

Now 1 is a root of the original equation $\sqrt{x} - 1 = 0$, since $\sqrt{1} - 1$ does equal 0. In this case, squaring both members of an equation results in an equation equivalent to the original equation.

Example 4. This example illustrates what may happen when both members of an equation are multiplied by a function containing the unknown.

Consider the equation

$$\frac{2x^2 - 6x + 4}{x^2 - x - 2} = 1.$$

If we clear this equation of fractions by multiplying both members of the equation by $x^2 - x - 2$, then

$$2x^2 - 6x + 4 = x^2 - x - 2.$$

Transposing and collecting gives

$$x^2 - 5x + 6 = 0;$$

factoring, we have

$$(x - 3)(x - 2) = 0;$$

then

$$x - 3 = 0 \qquad \text{or} \qquad x - 2 = 0.$$

Hence the roots are

$$x = 3 \qquad \text{and} \qquad x = 2.$$

However, let us check by substituting these values of x in the given equation to see whether they are roots of the given equation or not.

If $x = 3$, we have

$$\frac{2 \cdot 9 - 6 \cdot 3 + 4}{9 - 3 - 2} = \frac{18 - 18 + 4}{9 - 3 - 2} = \frac{4}{4} = 1.$$

If $x = 2$, we have

$$\frac{2 \cdot 4 - 6 \cdot 2 + 4}{4 - 2 - 2} = \frac{8 - 12 + 4}{4 - 2 - 2} = \frac{0}{0}.$$

We see that 3 satisfies the given equation and hence is a root, but 2 is an extraneous root for it does not satisfy the given equation. Had we simplified the fraction before clearing the equation of fractions, we would have avoided the extraneous root 2. Factoring both numerator and denominator, we have

$$\frac{2(x - 2)(x - 1)}{(x - 2)(x + 1)} = 1;$$

hence

$$\frac{2(x - 1)}{x + 1} = 1.$$

Clearing of fractions gives

$$2x - 2 = x + 1;$$

hence

$$x = 3.$$

After an equation is cleared of fractions, if the resulting equation has a root that is a zero of any of the denominators in the given equation, then this root cannot be a root of the given equation. That is, any root of the derived equation that is a zero of the least common denominator of both members of the given equation cannot be a root of the given equation.

Also, *when solving equations containing fractions, each fraction of the given equation should be reduced to its simplest form before clearing the equation of fractions.* If the numerator and denominator of any fraction in the equation contains a common factor and this common factor is not removed, then the equation cleared of fractions will contain roots which are zeros of the least common denominator.

Example 5. This example further illustrates what may happen when both members of an equation are multiplied by a function containing the unknown.

Whenever an equation is cleared of fractions by multiplying each member of the equation by a common denominator that is not the least common denominator of all the denominators, the resulting equation will always have extraneous roots.

Thus, suppose we clear the equation

$$\frac{1}{x-2} = \frac{3}{x^2-4}$$

of fractions by multiplying both members of the equation by $(x-2)(x^2-4)$ instead of the L.C.D. x^2-4. We would then have

$$x^2 - 4 = 3x - 6.$$

Transposing and collecting gives

$$x^2 - 3x + 2 = 0;$$

factoring, we obtain

$$(x-2)(x-1) = 0;$$

then

$$x - 2 = 0 \quad \text{and} \quad x - 1 = 0.$$

Hence the roots are

$$x = 2 \quad \text{and} \quad x = 1.$$

Let us check these results by substituting them in the given equation.
If $x = 1$,

$$\frac{1}{1-2} = \frac{3}{1-4} \quad \text{or} \quad -1 = -1.$$

If $x = 2$,

$$\frac{1}{2-2} = \frac{3}{4-4} \quad \text{or} \quad \frac{1}{0} = \frac{3}{0}.$$

Thus we see that 1 is a root of the given equation, but 2 is an extraneous root because 2 cannot be said to satisfy the given equation.

If we solve the equation $\frac{1}{x-2} = \frac{3}{x^2-4}$ correctly by clearing the equation of fractions by multiplying both members of the equation by the L.C.D. $x^2 - 4$, then

$$x + 2 = 3,$$

or

$$x = 1,$$

and thus we avoid the extraneous root.

When clearing an equation of fractions, each term of the equation should be multiplied by the ***least common denominator*** *of the entire equation, and not by any common denominator.*

An equation when cleared of fractions is equivalent to the original equation if no root of the derived equation is a zero of the multiplying function.

Example 6. Solve the equation

$$x^2 - x = 2x - 2.$$

Transposing and collecting gives

$$x^2 - 3x + 2 = 0;$$

factoring, we obtain

$$(x-1)(x-2) = 0;$$

then

$$x - 1 = 0 \quad \text{and} \quad x - 2 = 0.$$

Hence the roots are

$$x = 1 \quad \text{and} \quad x = 2.$$

Checking these results, we find that:

If $x = 1$, then $1 - 1 = 2 - 2$.
If $x = 2$, then $2^2 - 2 = 2 \cdot 2 - 2$.

Hence both 1 and 2 are roots of the given equation.

Now suppose we solve this equation by dividing both members of the equation by the common factor $x - 1$. Then the work would be as follows. Given

$$x^2 - x = 2x - 2. \tag{3}$$

Factoring both members

$$x(x - 1) = 2(x - 1), \tag{4}$$

and dividing both members by $x - 1$ give

$$x = 2. \tag{5}$$

Thus we see that the root 1 has been lost when we divided both members of the equation by the factor $x - 1$. In this case the derived equation is linear and cannot be equivalent to the original equation, which is a quadratic. Thus, the equation resulting from dividing both members by $x - 1$ is a defective equation.

Whenever both members of an equation are divided by a function of the unknown that is a common factor of both members, the resulting equation will be defective, and the derived equation will lack those roots of the original equation which are zeros of the dividing function.

In this example the derived Equation (5) lacks the root 1, which is a root of the original Equation (3). The root 1 is a zero of the dividing function $f(x) = x - 1$.

Example 7. The following will further illustrate what can happen when the operations that lead to redundant or defective equations are used. The student should analyze each step carefully. Suppose that

$$x = a;$$

multiplying by x gives

$$x^2 = ax;$$

subtracting a^2, we have

$$x^2 - a^2 = ax - a^2;$$

factoring both members

$$(x - a)(x + a) = a(x - a);$$

dividing by $x - a$, we get

$$x + a = a;$$

substituting a for x gives

$$a + a = a;$$

collecting gives

$$2a = a;$$

dividing by a, we obtain

$$2 = 1.$$

45. ALGEBRAIC SOLUTION OF LINEAR EQUATIONS IN ONE UNKNOWN

A linear equation is an equation of the first degree, and its standard form is

$$f(x) = a_0x + a_1 = 0. \tag{6}$$

The solution of this equation is obtained by subtracting a_1 from both members of the equation and then dividing through by a_0, the coefficient of x. Thus,

$$a_0x = -a_1, \tag{7}$$

$$x = -\frac{a_1}{a_0}. \tag{8}$$

Linear equations can be solved by collecting on the left-hand side of the equation all terms containing the unknown and collecting on the right-hand side of the equation all terms free of the unknown; then, after combining like terms, dividing both members of the equation by the coefficient of the unknown. To illustrate,

Example 1. Solve the equation $3x - 5 + a = bx + 1$ for x.

Solution. Given

$$3x - 5 + a = bx + 1;$$

transposing, we have

$$3x - bx = 1 + 5 - a;$$

collecting leaves

$$(3 - b)x = 6 - a;$$

dividing both members by $3 - b$ gives

$$x = \frac{6 - a}{3 - b}.$$

The following steps are helpful in solving equations containing fractions.

STEP 1. Rewrite the equation with all denominators factored and with the unknown arranged in descending powers.

STEP 2. Indicate all multiplication involved in clearing the equation of fractions by multiplying both members of the equation by the least common denominator.

STEP 3. Perform the indicated multiplication; if complicated, enclose the products in parentheses and remove parentheses in the next step.

STEP 4. Transpose all terms containing the unknown to the left-hand side of the equation and all terms free of the unknown to the right-hand side of the equation.

STEP 5. Combine like terms.

STEP 6. Solve for the unknown by dividing each member of the equation by the coefficient of the unknown.

STEP 7. Check by substituting the result in the original equation, not in any of the derived equations, to see if the result satisfies the given equation or is an extraneous root.

We shall illustrate the application of these seven steps in solving the following equation.

Example 2. Solve the equation

$$\frac{3x-2}{2x-6}-\frac{3-x^2}{9-x^2}=\frac{2x-3}{3x-9}-\frac{1}{6}.$$

Solution.

$$\frac{3x-2}{2(x-3)}-\frac{x^2-3}{(x-3)(x+3)}=\frac{2x-3}{3(x-3)}-\frac{1}{6}. \qquad \text{(Step 1)}$$

Multiplying each fraction by the L.C.D. $6(x-3)(x+3)$ and continuing the operations, we have

$$3(3x-2)(x+3)-6(x^2-3)$$
$$=2(2x-3)(x+3)-(x-3)(x+3), \qquad \text{(Step 2)}$$
$$3(3x^2+7x-6)-6(x^2-3)$$
$$=2(2x^2+3x-9)-(x^2-9), \qquad \text{(Step 3)}$$
$$9x^2+21x-18-6x^2+18$$
$$=4x^2+6x-18-x^2+9,$$
$$21x-6x=9-18, \qquad \text{(Step 4)}$$
$$15x=-9, \qquad \text{(Step 5)}$$
$$x=-\tfrac{9}{15}=-\tfrac{3}{5}. \qquad \text{(Step 6)}$$

Check:

$$\frac{-\frac{9}{5}-2}{-\frac{6}{5}-6}-\frac{3-\frac{9}{25}}{9-\frac{9}{25}}=\frac{-\frac{6}{5}-3}{-\frac{9}{5}-9}-\frac{1}{6}, \qquad \text{(Step 7)}$$
$$\frac{-19}{-36}-\frac{66}{216}=\frac{-21}{-54}-\frac{1}{6},$$
$$\tfrac{19}{36}-\tfrac{11}{36}=\tfrac{7}{18}-\tfrac{3}{18},$$
$$\tfrac{8}{36}=\tfrac{4}{18}.$$

46. GRAPHICAL SOLUTION OF LINEAR EQUATIONS

If a linear equation contains only numerical coefficients, then the root of this equation can be obtained by graph.

Example 1. Solve the equation $2x + 5 = 0$ graphically.

Solution. Set $y = 2x + 5$ and graph the equation $y = 2x + 5$. Then the root of the equation $2x + 5 = 0$ is the x-intercept of the graph of $y = 2x + 5$. The graph of $y = 2x + 5$ is drawn in Fig. 14 by plotting the accompanying table of data. From this graph we see that the x-intercept of the graph of the function $y = 2x + 5$ is $-2\frac{1}{2}$, which is the root of the given equation $2x + 5 = 0$.

x	y
−5	−5
−4	−3
−3	−1
−2	1
−1	3
0	5
1	7

When an equation contains more than the one letter, and the unknown is not specified, we shall in this course assume the unknown to be represented by a letter including or following ***p***.

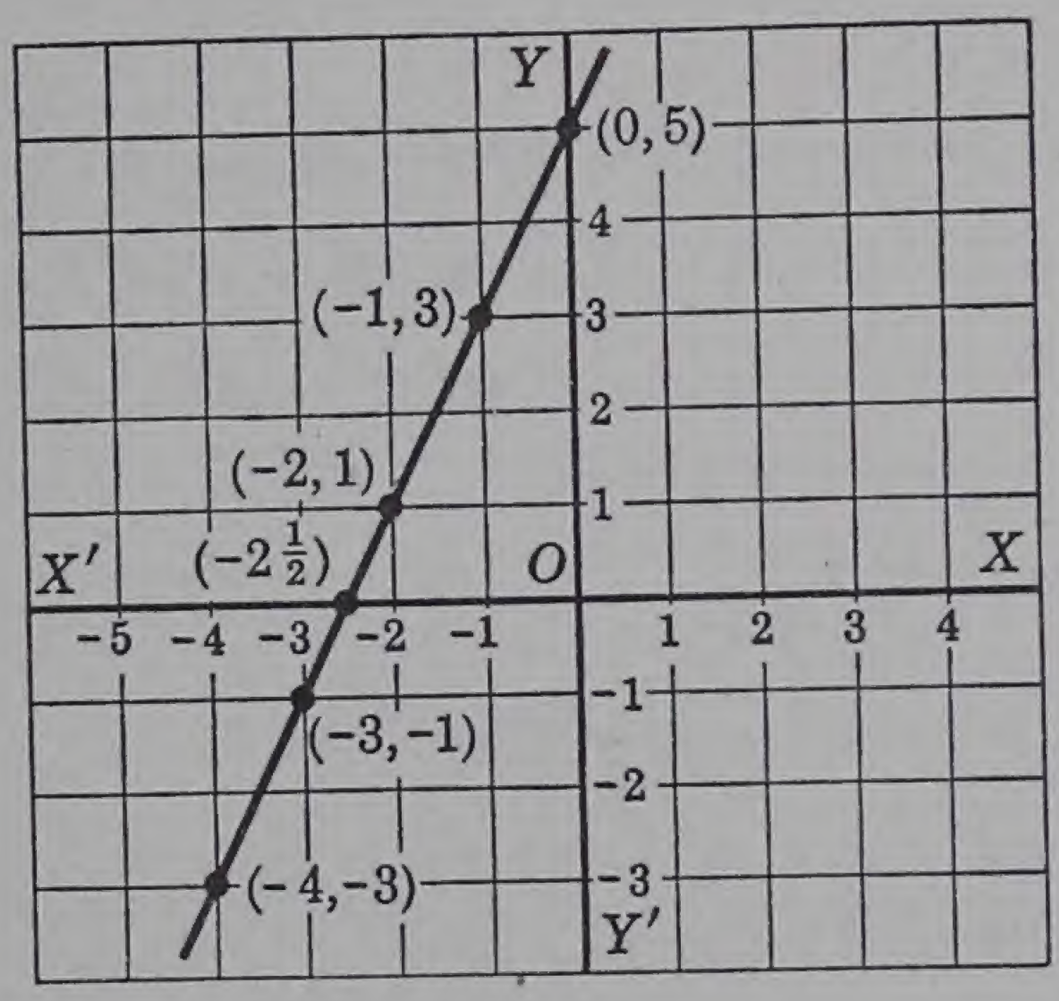


FIG. 14.

QUESTIONS

1. Define algebraic equation.
2. What is meant by solution? By root?
3. What is meant by equivalent equations?
4. What operations on an equation will always give an equivalent equation?
5. What operations on an equation may give an equation that is not equivalent to the original equation?
6. How are equations cleared of fractions?
7. What is meant by an equation in its standard form?
8. Define linear equation.
9. Give two methods for solving linear equations.

EXERCISES

Write the equations in Exercises 1 and 2 in the standard form and give the values of n, a_0, a_1, a_2, and so on.

1. (a) $3x^2 - 2x - 5 = 0$; (b) $3x^3 - 5x + 7 = 0$;
(c) $x^5 - 5x^4 - 2x - 1 = 0$; (d) $x^4 - 3x = 0$;
(e) $5x^6 = 12$; (f) $3x^2 - 5x = 7x + 1$.

2. (a) $9x^3 - 4x^2 = 3x - 2$; (b) $x^4 + (2x - 3)^2 = 5$;
(c) $x^3 - 3x^2 + ax^2 - 3x + b = 0$;
(d) $2x^2 - mx + 3m + x - 2 = 0$;
(e) $\dfrac{x - 1}{x + 2} = 5$; (f) $\dfrac{3x + 2}{2x - 1} = 1 - \dfrac{x}{x - 2}$.

3. Solve the following equations and show the solution by graphs:
(a) $x - 5 = 0$; (b) $3x + 2 = 0$; (c) $5x = 10$; (d) $6x = 8$.

4. Is 2 a root of the equation $x^2 - 3x + 4 = 0$? Why?

5. Is 1 a root of the equation $x^7 + 5x^6 - 8x^2 + 2 = 0$? Why?

6. Which of the numbers 1, 2, -1, -2, 0 are roots of the equation $x^4 - 2x^3 + x - 2 = 0$?

Solve each of the following equations (Exercises 7 through 44).

7. $5x - 2 = 7x + 6$.

8. $6x - 4 = x + 6 - 10$.

9. $2x + \dfrac{3}{4} = \dfrac{7x}{3}$.

10. $\dfrac{2x - 3}{3} - \dfrac{3x - 2}{2} = x - 2$.

11. $\dfrac{2x - 3}{5} - \dfrac{6x - 7}{3} = 4$.

12. $\dfrac{2x - 1}{2} - \dfrac{3x - 1}{3} = \dfrac{6x - 1}{6} - 2$.

13. $\dfrac{x - 3}{3} - \dfrac{x^2 - 3x}{3x - 2} = \dfrac{2}{9}$.

14. $\dfrac{2}{x + 1} - x = \dfrac{1 - 2x}{2} - 3$.

15. $\dfrac{6y}{y - 1} - \dfrac{6y - 2}{y + 1} = \dfrac{2}{y^2 - 1}$.

16. $\dfrac{3}{x - 3} - \dfrac{x - 3}{x + 3} + \dfrac{x^2}{x^2 - 9} = 0$.

17. $\dfrac{2x^2 - 3x + 1}{2x - 1} = \dfrac{3x - 1}{3}$.

18. $\dfrac{x - 2}{x + 2} - 2 + \dfrac{x + 3}{x - 3} = 0$.

19. $\dfrac{5}{3x^2 - x} + \dfrac{2}{3x} = \dfrac{2}{3x - 1}$.

20. $\dfrac{2x + 1}{2(x + 1)} - \dfrac{3x - 1}{3(x - 1)} = \dfrac{1 - 6x}{6 - 6x^2}$.

21. $\dfrac{3x - 4}{2x - 1} - 3 = \dfrac{5 - x}{x - 2} + \dfrac{x}{1 - 2x}$.

22. $\dfrac{x + 1}{1 - x} - \dfrac{1 - x}{x + 1} = \dfrac{5}{x^2 - 1}$.

23. $\dfrac{2x-1}{(x-2)^2}+\dfrac{3x+4}{4-x^2}=-\dfrac{1}{2+x}$.

24. $\dfrac{(x-2)(x+3)}{(x+2)(x-3)}=\dfrac{2+3x}{2+x}-\dfrac{3-2x}{3-x}$.

25. $\dfrac{3x-2}{x^2-4}-\dfrac{2x-3}{(2-x)^2}=\dfrac{3}{4+2x}+\dfrac{1}{4-2x}$.

26. $\dfrac{2}{x+2}+\dfrac{2}{x^2-4}=\dfrac{x}{x^2-4}$.

27. $\dfrac{1}{x-1}-\dfrac{1}{x+4}=\dfrac{1}{x-3}-\dfrac{1}{x+2}$.

28. $\dfrac{x}{x-1}+\dfrac{3x}{2x-2}-\dfrac{5x}{3x-3}=\dfrac{3}{5}$.

29. $\dfrac{2}{x-3}-\dfrac{3}{x^2-9}=\dfrac{x}{x^2-9}$.

30. $\dfrac{x}{x-1}-\dfrac{2x-1}{2x-2}+\dfrac{x+2}{3x-3}=\dfrac{3}{x-1}-\dfrac{2}{3}$.

31. $3a-6x=5a$.

32. $cx+d=a$.

33. $ax+bx=c$.

34. $(a+b)x=ax+b$.

35. $(x+a)(x-b)=(x+a-b)^2$.

36. $\dfrac{a^2b-x}{a}-\dfrac{b^2c+x}{b}-\dfrac{c^2a-x}{c}=0$.

37. $\dfrac{x-a}{a}-\dfrac{x-b}{b}=\dfrac{x-b}{a}+\dfrac{x-a}{b}$.

38. $\dfrac{ax+b}{ax-b}+\dfrac{8b^2}{b^2-a^2x^2}-\dfrac{ax-b}{ax+b}=0$.

39. $\dfrac{x+a}{a}-x+a=\dfrac{x-b}{b}+x-b$.

40. $\dfrac{1}{a-b}-\dfrac{a-b}{x}=\dfrac{1}{a+b}-\dfrac{a+b}{x}$.

41. $\dfrac{1}{x}-\dfrac{1}{a}=\dfrac{1}{b}-\dfrac{1}{x}$.

42. $\dfrac{1}{a}+\dfrac{a}{x+a}=\dfrac{x+a}{ax}$.

43. $\dfrac{ab}{x}-ab=c+\dfrac{d}{x}$.

44. $\dfrac{x-a}{x+a}-\dfrac{x-b}{x+b}=\dfrac{2(b-a)}{x}$.

In each of the formulas in Exercises 45 through 50 solve for each of the indicated letters in terms of the remaining letters in the formula.

45. Solve for a, n, and d in the formula $l = a + (n - 1)d$.

46. Solve for a, n, and l in the formula $s = \frac{n}{2}(a + l)$.

47. Solve for P, i, and n in the formula $A = P(1 + in)$.

48. Solve for h, a, and b in the formula $A = \frac{1}{2}h(a + b)$.

49. Solve for a, r, and l in the formula $s = \dfrac{rl - a}{r - 1}$.

50. Solve for R, R_1, and R_2 in the formula $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$.

47. PROBLEMS

There are certain types of problems in which there are no equations given, but from the statement of the problem an equation can be formed that will give a relation connecting the unknown with the known parts. The following steps will aid us to set up the equations connecting the known parts with the unknown parts.

STEP 1. Read the problem very carefully until you have a clear idea of *what is given, what is to be found, and what things are equal.*

STEP 2. *Represent each unknown quantity by some letter and specify the units representing this quantity.*

Do not say "let x equal the amount added," when you should say "let x equal the number of quarts of alcohol added to the original mixture," or, as some prefer to say, "let x equal the **number representing** the number of quarts of alcohol added to the original mixture." Do not merely say "let x equal the speed of the train," when you should say "let x equal the speed of the train in miles per hour," or "feet per second," as the case may be.

In most cases it is advisable to let the unknown or unknowns represent the quantity or quantities to be determined.

STEP 3. *State carefully and completely in terms of proper units what is being equated.*

Since equations are expressions of equality, two representations for the same thing must be found to set equal to each other. That is, if we have the number of years in Mary's present age for the left-hand member of the equation, we must have the number of years in Mary's present age for the right-hand member of the equation. If we have the number of pounds of pure silver in the final mixture for the left-hand member of the equation, we must

have the number of pounds of pure silver in the final mixture for the right-hand member of the equation.

STEP 4. Solve the resulting equation.

STEP 5. Check the results by determining whether or not they satisfy the conditions of the given problem. It is not sufficient that the results satisfy the original equation, because the equation may not be set up correctly. Also, some of the roots may be impossible, as for example, -7 could not be an answer if it meant -7 boys.

Problem 1. How much pure alcohol must be added to 10 quarts of a 40% solution of alcohol to give a 60% solution of alcohol?

Solution. Let x equal the number of quarts of pure alcohol to be added.

Equate the number of quarts of pure alcohol in the final mixture.

Since the final mixture contains $x + 10$ quarts in all, we may write the following statements:

$.40(10)$ = the number of quarts of pure alcohol in the 40% solution,
x = the number of quarts of pure alcohol added,
$.60(10 + x)$ = the number of quarts of pure alcohol in the final mixture.

Therefore, we may write the equation:

$$.40(10) + x = .60(10 + x).$$

Hence,

$$4 + x = 6 + .6x;$$

transposing, we have

$$x - .6x = 6 - 4;$$

collecting gives

$$.4x = 2;$$

dividing by .4, we find that

$$x = 5$$

is the number of quarts of pure alcohol to be added.

Problem 2. One man drives an automobile 35 miles per hour and has a start of 4 hours over a second man who drives 45 miles per hour in the same direction. In how many hours will the second man overtake the first man?

Solution. Let x equal the number of hours the second man traveled.

Equate the distance in miles each car traveled. Thus:

$$\begin{aligned} 45x &= 35(x + 4) \\ &= 35x + 140; \end{aligned}$$

collecting, we have

$$10x = 140;$$

hence,

$$x = 14$$

is the number of hours the second man traveled.

We might solve this problem in the following manner.

Let y equal the number of miles each man traveled.

Equate the number of hours the first man traveled. Thus:

$$\frac{y}{35} = \frac{y}{45} + 4;$$

L.C.D. is 315, so

$$9y = 7y + 1{,}260;$$

collecting, we have

$$2y = 1{,}260;$$

hence,

$$y = 630$$

is the number of miles each man traveled.

Therefore,

$$\frac{y}{35} = 18$$

is the number of hours the first man traveled, and

$$\frac{y}{45} = 14$$

is the number of hours the second man traveled.

The first equation for each method is apparent, since the first man traveled 4 hours longer than the second man traveled.

Problem 3. A can do a piece of work in 8 days and B can do the same work in 5 days; in how many days will it take both to do all the work when working together?

Can we equate the number of days it takes both to do all the work? Why?

Solution. Let x equal the number of days it takes both to do all the work.

Equate the part of all the work they both can do in 1 *day.* Thus:

$$\frac{1}{x} = \frac{1}{8} + \frac{1}{5};$$

collecting, we have

$$\frac{1}{x} = \frac{13}{40};$$

hence,

$$x = \frac{13}{40} = 3\,\frac{1}{13}$$

is the number of days for both to do all the work.

The equation is apparent, since

$$A \text{ can do } \frac{1}{8} \text{ th of all the work in 1 day,}$$

and

$$B \text{ can do } \frac{1}{5} \text{ th of all the work in 1 day,}$$

and

$$\text{both can do } \frac{1}{x} \text{ th of all the work in 1 day.}$$

PROBLEMS

1. The sum of two numbers is 45, and the one number is 4 times the other number. What are the numbers?

2. If the sides of a square are increased by 5 feet, the area will be increased by 175 square feet. What were the original dimensions of the square?

3. One man walks 4 miles per hour and a second man walks 6 miles per hour. If they walk in opposite directions, how long will it take for them to be 25 miles apart?

4. In making a trip a man averaged 35 miles per hour going, but on the return trip he averaged only 30 miles per hour. If it took $9\frac{3}{4}$ hours to make the trip, how far from home did he go?

5. A father is 42 years old; the son is 8 years old. When will the father be 3 times as old as the son?

6. A father is three times as old as his son, but 15 years hence he will be only twice as old as his son. What are the present ages of father and son?

7. A boy buys a number of marbles at the rate of 3 for a dime. He sells one-third of them at 4 for a quarter and the remainder at 2 for five cents, and makes a profit of twenty cents on the transactions. How many marbles did he buy?

8. A man packing breakable articles gets 4 cents for each article packed but must pay 10 cents for each article he breaks. How many articles must he pack each day to make \$5 per day if he breaks 6 articles a day? What per cent of the articles are broken?

9. One boy can pick 1 gallon of berries in 2 hours, and a second boy can pick 1 gallon in $2\frac{1}{2}$ hours. How long will it take the two boys together to pick 6 gallons of berries? How many gallons of berries can they pick in 5 hours?

10. It takes A 40 hours to paint a house and B 30 hours to paint the same house. If they paint the house together, what will be each one's wages if A receives \$1.05 per hour and B receives \$1.40 per hour?

11. At what time between four and five o'clock are the two hands of a clock together?

12. At what time between four and five o'clock do the two hands of a clock form a straight line?

13. A fighter plane at 1:15 p.m. observes a bomber 10 miles away. If the fighter plane travels 450 miles per hour and the bomber travels 350 miles per hour, when and where will the fighter overtake the bomber?

14. Two automobiles start together and travel in the same direction. The one travels 45 miles in 70 minutes and the other travels 30 miles in 55 minutes. When will the one automobile be 30 miles ahead of the other?

15. A tank contains 10 gallons of 5% salt brine. How much water must be evaporated so that the remaining portion will be 8% salt?

16. A container contains a 75% mixture of water and alcohol, a second container contains a 50% mixture of water and alcohol. What proportions must be taken from each container to give a 60% mixture?

17. How much alcohol must be added to a 50% solution to give 25 quarts of 60% solution?

18. A smelter has 640 pounds of a 50% silver alloy and 200 pounds of a 20% silver alloy. How much of the 50% silver alloy must be mixed with all the 20% silver alloy to give a 30% silver alloy?

19. A can do a piece of work in one-third the time B can; B can do it in three-fourths the time C can; all together they can do it in 12 days. How long will it take each alone to do the work?

20. A and B together can do a piece of work in 10 days; A and C together can do the work in 12 days; and B and C together can do it in 15 days. In how many days can they do the work if they all work together?

21. A can do a piece of work in $12\frac{1}{2}$ days; B can do the same amount of work in $\frac{2}{5}$ of the time A can; and C can do the work in $3\frac{1}{3}$ times the time it takes all three to do the work. How long does it take C to do the work alone?

22. A can do a certain amount of work in $\frac{3}{4}$ the time that B can; B can do the same amount in $\frac{2}{3}$ the time C can. The three together can do the work in 24 days. How long will it take each of them to do the work alone?

23. An aircraft carrier is traveling toward shore at the rate of 25 miles per hour. At 3:30 p.m. when the carrier is 150 miles from shore, a scout plane leaves the carrier and flies to the shore and back to the carrier. If the scout plane travels at 200 miles per hour, at what time will the plane reach the carrier?

24. An airplane, flying 100 miles per hour, passes over an automobile traveling in the same direction on a straight road. One hour later it passes a second automobile traveling in the opposite direction. If the two automobiles pass each other 45 minutes later and are traveling at the same speed, find their speed.

25. If $\frac{5}{8}$ of the members of a club of 120 members are in favor of a certain proposition, how many new members, all favoring the proposition, must be added to the club so as to have $\frac{3}{4}$ of the members favoring the proposition?

26. In a group of 55 persons there are $\frac{5}{6}$ as many men as women, and there are 2 more women than twice the number of children. How many of each are there?

27. In 12 years a boy will be 3 times as old as he was 10 years ago. How old is he?

28. If *A*'s wealth is doubled, he will then have as much more than \$6,000 as he now has less than \$6,000. Find his wealth.

29. In a purse there are a certain number of nickels, three-fourths as many dimes, two-thirds as many quarters, one-fourth as many half dollars, and one-third as many dollars. If the total value in the purse is \$9, how many pieces are there of each kind?

30. A merchant sold a suit of clothes for \$45, which was at a profit of one-half of the cost of the suit. What was the cost of the suit?

31. If each day a man spends one-third of the money he has at the beginning of the day and if he has \$8 at the end of the third day, how much did he have to start with?

32. In a 100-yard dash *A* beats *B* by 3 feet and *C* by 5 feet. By how much does *B* beat *C*?

33. A grocer has some coffee worth 25 cents per pound and some worth 40 cents per pound. What proportion of each must be used to give a mixture worth 30 cents per pound?

34. If the minute hand of a clock passes the hour hand every 65 minutes, what is the per cent of error of the clock? Is the clock fast or slow?

35. A rabbit takes 5 leaps to a dog's 4; but 2 of the dog's leaps are equivalent to 3 of the rabbit's. If the rabbit has a start of 45 of his leaps, how many leaps must the dog take to catch the rabbit?

36. A rabbit takes 5 leaps to a dog's 3; but 2 of the dog's leaps are equivalent to 3 of the rabbit's. If the rabbit has a start of 50 of his leaps, will the dog ever catch the rabbit? After the dog makes 100 leaps, how many rabbit leaps is the rabbit ahead of the dog?

37. How much cream testing 18% butter fat must be added to 1,000 pounds of milk testing 3% butter fat to give milk containing 3.5% butter fat?

38. The radiator of an automobile holds 14 quarts. At the beginning of cold weather 2 quarts of alcohol were added, later 2 more quarts of alcohol were added, still later a 40% mixture was desired. How much alcohol was needed the last time? (Assume that there is no loss by evaporation.)

39. The sum of two numbers is s, and the one number is t times the other number. What are the numbers?

40. Two automobiles start at the same place and travel in opposite directions at speeds of V_1 and V_2 miles per hour, respectively. How long will it take for them to be m miles apart?

41. A father is f years old; the son is s years old. When will the father be n times as old as the son?

42. In n years a boy will be t times as old as he was m years ago. How old is he?

43. If A requires a days to do a piece of work and B requires b days to do the same work; how long will it take both to do the work together?

44. In making a trip a man averaged V_1 miles per hour going, but returning he averaged V_2 miles per hour. If he was t hours on the road, how far from home did he go?

45. In a 100-yard dash A beats B by b feet and C by c feet. By how much will B beat C?

46. How many pounds of cream testing $c\%$ butter fat must be added to n pounds of milk testing $m\%$ butter fat to give milk testing $t\%$ butter fat?

47. How much tea selling at a cents per pound must be mixed with tea selling at b cents per pound to make n pounds of mixture to sell at c cents per pound (a greater than b)?

48. A and B travel toward each other from points d miles apart, A at the rate of a miles per hour and B at the rate of b miles per hour. When and where will they meet if B starts c hours after A starts?

49. If oranges cost a cents a dozen, how many oranges can be bought for d dollars?

50. If a certain article sells for d dollars a dozen, how many articles can be bought for c cents?

48. LEVER PROBLEMS

A **lever** is defined to be a rigid bar that is capable of a motion of rotation about a fixed point called the fulcrum. In Fig. 15 the fulcrum is the point F. If a weight W is attached to the lever at a distance d from the fulcrum, then d is called the **lever arm** of the

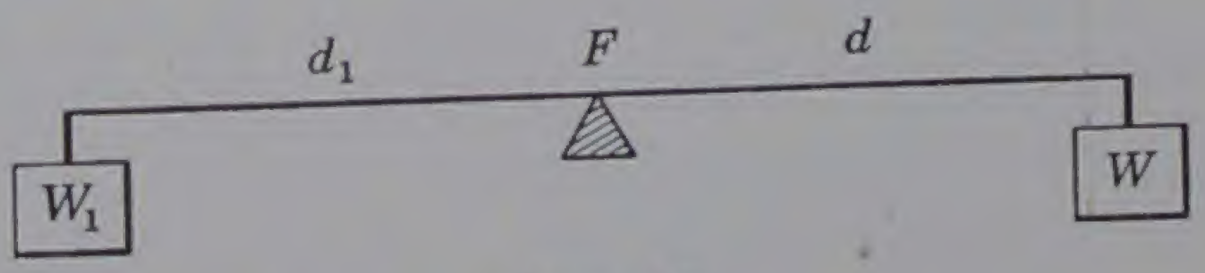


FIG. 15.

weight W, and the product Wd of the weight and the lever arm is called the **moment** of the force W about the fulcrum F.

FUNDAMENTAL PRINCIPLE. *A lever is in equilibrium when, and only when, the sum of the moments of all forces acting on one side of the fulcrum equals the sum of the moments of all forces acting on the other side of the fulcrum.* Thus, the lever in Fig. 15 is in equilibrium if $Wd = W_1d_1$ (assuming the lever to be weightless).

The fundamental principle may also be stated in the following form. *A lever is in equilibrium when, and only when, the sum of the moments of all forces acting to produce clockwise rotation about any given point is equal to the sum of the moments of all forces acting to produce counterclockwise rotation about the same point.*

Example. A lever 27 feet long has 40 pounds fastened at one end, 20 pounds fastened at the other end, and 30 pounds fastened 15 feet from the 40-pound weight. Where must the fulcrum be placed if the lever is in equilibrium?

Solution. *First Method.* Let x equal the number of feet the fulcrum is from the 40-pound weight. Applying the first form of the fundamental principle, and equating the moment on the left side of the fulcrum to the sum of the two moments on the right side of the fulcrum, we have

$$\begin{aligned} 40x &= 30(15 - x) + 20(27 - x) \\ &= 450 - 30x + 540 - 20x. \end{aligned}$$

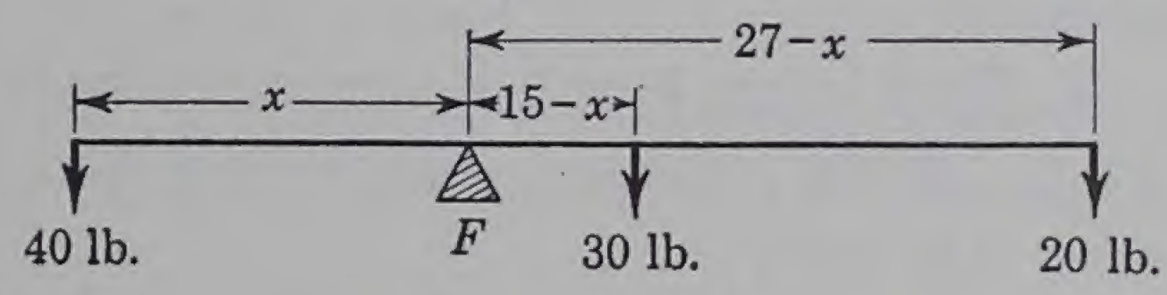
Collecting, gives

$$90x = 990.$$

Solving for x, we get

$$x = 11$$

feet from the 40-pound weight.



FIRST METHOD.

Solution. *Second Method.* Applying the second form of the fundamental principle, using the left end of the lever as the point of rotation, and equating the counterclockwise moment to the sum of the two clockwise moments, we have

$$\begin{aligned} 90x &= 30 \cdot 15 + 20 \cdot 27 \\ &= 450 + 540. \end{aligned}$$

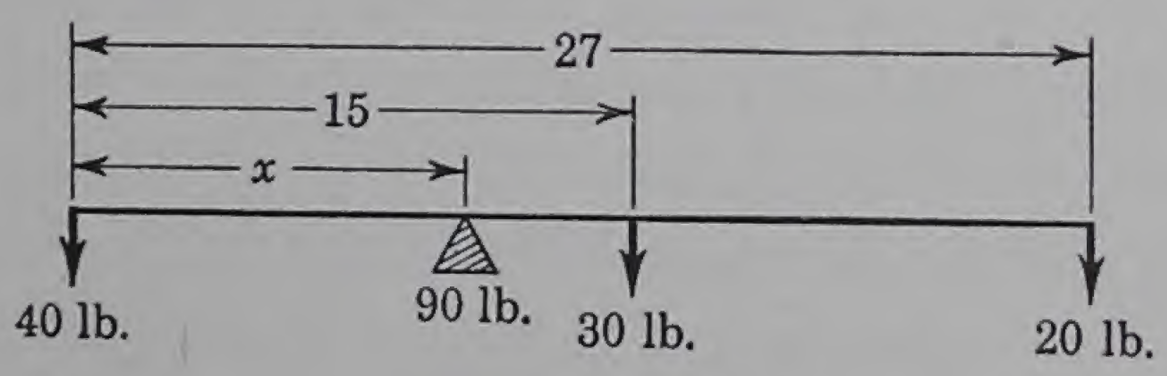
Collecting, we obtain

$$90x = 990.$$

Solving for x gives

$$x = 11$$

feet from the 40-pound weight.



SECOND METHOD.

Observe that the force of the fulcrum is

directed upward and that this force is the sum of the three weights, or 90 pounds.

EXERCISES

1. Where should the fulcrum be placed on a seesaw 10 feet long in order that a child weighing 80 pounds at the one end balances a child weighing 120 pounds at the other end?

2. What is the heaviest load that can be moved by a man exerting a force of 180 pounds on the longer arm of an 8-foot lever, if the fulcrum is 2 feet from the other end?

3. What is the heaviest load that can be moved by a man exerting a force of 180 pounds on the longer arm of a plank which is 8 feet long and weighs 8 pounds per foot, if the fulcrum is 2 feet from the other end?

4. A child weighing 36 pounds wishes to seesaw on a plank which is 12 feet long and weighs 6 pounds per foot. Where must the fulcrum be placed if child and plank are in balance?

5. Two boys weighing 156 and 60 pounds respectively are on opposite ends of a seesaw which is 12 feet long and weighs 6 pounds per foot. Where must the fulcrum be placed if they balance each other?

6. Two children weighing 40 and 60 pounds, respectively, are on opposite ends of a 10-foot board with the fulcrum at the center. Where must a 40-pound weight be fastened to balance the board?

7. A weight of 80 pounds suspended on a pole 5 feet long is carried by two boys, one at each end of the pole. Where should the load be hung in order that one boy may carry 32 pounds and the other may carry 48 pounds?

8. Two boys, A and B, are carrying 60 pounds hung from a pole 5 feet long. If the load is 2 feet from A and 3 feet from B, what weight does each boy carry?

9. A lever 12 feet long has 40 pounds at one end, 60 pounds at the other end, and 30 pounds hung 4 feet from the 40-pound weight. Where is the fulcrum if the lever is in equilibrium?

CHAPTER VIII

Systems of Simultaneous Equations

METHODS FOR SOLVING SIMULTANEOUS EQUATIONS

1. *Elimination by Addition and Subtraction.*
2. *Elimination by Substitution.*
3. *Determinants.*
4. *Graphs.*

49. GENERAL DISCUSSION

If one variable y is expressed directly as a function of a second variable x, then y is called an **explicit function** of x and is written $y = f(x)$. For example, in the equation

$$y = f(x) = \tfrac{1}{3}(2x - 5)$$

y is an explicit function of x. If, on the other hand, the variable y is defined in terms of the variable x and their relationship is given by means of a single function of the two variables, then each variable is an **implicit function** of the other variable and the equation is written as

$$f(x, y) = 3x - 2y + 5 = 0;$$

then y is said to be an implicit function of x, or x is an implicit function of y.

One equation in two variables has an unlimited number of pairs of values of the variables that will satisfy the given equation. If, however, we consider two nonequivalent equations of two variables each, there will be only a finite number of pairs of values (real or imaginary) of the unknowns that will satisfy both equations, although there is an unlimited number of pairs of values of the two variables that will satisfy each equation separately.

To illustrate, in $x + y = 5$ there is an unlimited number of pairs of values of x and y whose sum is 5, and in $x - y = 3$ there is an unlimited number of pairs of values of x and y whose difference

is 3. However, there is but one pair of values of x and y whose sum is 5 and whose difference is 3. This pair of values is $x = 4$ and $y = 1$.

We speak of x and y in each of the separate equations as variables, but when we consider the two equations as a system of equations, we speak of x and y as unknowns, for in the system they have fixed values.

To solve a system of **n** *equations in* **n** *unknowns, is to find all the sets of values of the unknowns, each set of which will satisfy each of the equations of the given system. The numbers composing a set are called a solution and these numbers are constants, real or imaginary.*

If a system of equations has a *finite number of solutions*, the equations of the system are said to be **consistent** and **independent.**

If a system of equations is satisfied by an *unlimited number of solutions*, the equations of the system are said to be **consistent** and **dependent.**

If no set of values (real or imaginary) exists that will satisfy each equation of the system, the equations of the system are said to be **inconsistent.**

Since the graph of an equation of two variables lies in a plane and is the curve that contains all the points whose coördinates satisfy the equation and no other points, it follows that the coördinates of any points of intersection of the two graphs of a system of two equations in two unknowns must satisfy each equation of the system and therefore must be a solution of the system.

We shall state the following without proof:

To solve a system of linear equations in two unknowns, it takes two and only two independent equations in the two unknowns to determine all the pairs of values of the two unknowns that will satisfy both equations of the system.

Likewise, *it takes a system of* **n** *independent equations in* **n** *unknowns to determine all the sets of values of the* **n** *unknowns that will satisfy all* **n** *equations.*

50. SYSTEMS OF TWO LINEAR EQUATIONS IN TWO UNKNOWNS

Since the derivative, or the slope of the graph, of the linear equation

$$y = mx + b$$

is

$$\frac{dy}{dx} = m,$$

where m is a constant, it follows that the graph of $y = mx + b$ is a straight line with the constant slope m. That is, *the graphs of all linear equations in two variables are straight lines.*

The study of systems of linear equations dates back to the Egyptians, centuries before Christ.

We shall give four methods for solving a system of two linear equations in two unknowns. These methods are:

1. *Method of elimination by addition and subtraction.*
2. *Method of elimination by substitution.*
3. *Method of determinants.*
4. *Method of graphs.*

Method of Elimination by Addition and Subtraction

It is possible to solve a system of two linear equations in two unknowns by eliminating one of the two unknowns between the two equations and then solving the resulting equation in the other unknown. This can be done by selecting two constants so that if each term of the first equation is multiplied by the one constant and each term of the second equation is multiplied by the other constant, the resulting equations will have the same numerical value for the coefficient of one of the unknowns; then by adding or subtracting the members of the two resulting equations, depending on the signs of the common coefficient, the one unknown is eliminated from the resulting equation. Solve this last equation for the other unknown. To illustrate, let us solve the system of equations in the following example.

Example 1. Solve

$$2x - 3y = 13, \tag{1}$$
$$5x + 6y = -8. \tag{2}$$

Solution. *By Addition and Subtraction.* To eliminate y we need only to multiply each term of Equation (1) by 2 and add the members of the resulting equation to the members of Equation (2). Doing this, we have

$$\begin{array}{rcr} 4x - 6y = & & 26 \\ 5x + 6y = & & -8 \\ \hline 9x & = & 18 \end{array}$$

Solving this equation for x gives

$$x = 2.$$

It is possible to solve for y, either by substituting this value of x in Equation (1) or (2), or by eliminating x by multiplying each term of Equation (1) by 5 and each term of Equation (2) by 2, and by subtracting the members of the two resulting equations. Using the latter method to eliminate x, we have

$$\begin{array}{rr} 10x - 15y = & 65 \\ 10x + 12y = & -16 \\ \hline -27y = & 81 \end{array}$$

Solving this equation for y gives

$$y = -3.$$

Hence, the solution of the given system of Equations (1) and (2) is $x = 2$, and $y = -3$. This solution can also be written in the form $(2, -3)$.

Check: Substituting 2 for x and -3 for y in Equation (1), we have

$$2(2) - 3(-3) = 13,$$
$$4 + 9 = 13.$$

or

Substituting the values in Equation (2), we have

$$5(2) + 6(-3) = -8,$$

or

$$10 - 18 = -8,$$

which indicates that the solution is correct.

Method of Substitution

To solve a system of two linear equations in two unknowns by substitution, solve either of the two given equations for the one unknown in terms of the other unknown and then substitute this value in the other equation. The resulting equation will be linear and will contain but the one unknown, and hence this equation can be solved for the one unknown. To illustrate, let us now solve the foregoing system of equations by the method of substitution.

Example 2. Solve

$$2x - 3y = 13, \quad (1)$$
$$5x + 6y = -8. \quad (2)$$

Solution. *By Substitution.* Suppose we solve Equation (1) for x in terms of y; then

$$x = \frac{13 + 3y}{2}.$$

Substituting this value of x in Equation (2), we have

$$\frac{5(13 + 3y)}{2} + 6y = -8.$$

Simplifying gives

$$65 + 15y + 12y = -16,$$

or

$$27y = -81;$$

solving for y, we get

$$y = -3.$$

Since $x = \frac{13 + 3y}{2}$, then

$$x = \frac{13 + 3(-3)}{2} = 2.$$

Therefore, the solution of the system is $x = 2$, $y = -3$.

Check: Now $x = 2$ and $y = -3$ are such that they satisfy Equation (1), but one should always check to see whether or not the results satisfy the other equation. It may happen that an error was made in the work when solving for y, and hence we may have a pair of values that will satisfy the one equation but not the other. Checking in (2), we have

$$5 \cdot 2 + 6(-3) = -8,$$

or

$$10 - 18 = -8.$$

Method of Determinants

Let us first solve by addition and subtraction the system composed of the two general linear equations

$$a_1x + b_1y = c_1, \tag{3}$$
$$a_2x + b_2y = c_2. \tag{4}$$

To eliminate y, we multiply each term of Equation (3) by b_2 and each term of Equation (4) by b_1. This gives the following two equations whose members may be subtracted from each other

$$\begin{aligned} a_1b_2x + b_1b_2y &= c_1b_2 \\ a_2b_1x + b_2b_1y &= c_2b_1 \\ \hline (a_1b_2 - a_2b_1)x &= c_1b_2 - c_2b_1 \end{aligned}$$

Solving the resulting equation for x gives

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}. \quad (a_1b_2 - a_2b_1 \neq 0.) \tag{5}$$

To eliminate x we multiply each term of Equation (3) by $-a_2$ and each term of Equation (4) by a_1. Then we have the following two equations whose members we add:

$$\begin{array}{r} -a_2a_1x - a_2b_1y = -a_2c_1 \\ a_1a_2x + a_1b_2y = a_1c_2 \\ \hline (a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1 \end{array}$$

Solving the sum for y, we obtain

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \quad (a_1b_2 - a_2b_1 \neq 0.) \qquad (6)$$

Since

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1} \qquad \text{and} \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

are the solutions of the two general linear equations (3) and (4), we may consider them as the formula for the solution of all systems of two linear equations in two unknowns, where proper values are to be substituted for the a's, b's, and c's. The formula in this form may be a little difficult to remember, but we can write it by employing the following symbolic form that is easier to remember. Thus:

$$a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}. \qquad (7)$$

The symbol on the right side of Equation (7) is called a **determinant of order two.** This symbol must always mean the product a_1b_2 minus the product a_2b_1.

The denominators of the values of x and y given by equations (5) and (6) are equal and are given by Equation (7). Applying this symbolic notation to the numerators of x and y, we have the following:

$$c_1b_2 - c_2b_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \qquad (8)$$

$$a_1c_2 - a_2c_1 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \qquad (9)$$

which conform to the definition of the symbol given in Equation (7).

In the preceding discussion the solution of the system of the two general linear equations

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2,$$

is shown to be

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \tag{10}$$

By using the notation of determinants, this solution can be written as the ratio of two determinants, thus,

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \tag{11}$$

provided the determinant in the denominator is not equal to zero. The determinant in the denominator is called the **determinant of the system** of equations.

Observe that the two given equations are written with the terms containing the unknowns in the left-hand members and the constant terms in the right-hand members. The determinant of the system is a square array of the coefficients of the two unknowns, of which the first column consists of the coefficients of the one unknown and the second column consists of the coefficients of the other unknown. The value of each unknown has the determinant of the system for its denominator and the determinant in the numerator is like the determinant in the denominator except that the column containing the coefficients of the unknown for which we are solving is replaced by the respective constants on the right-hand side of the equality signs.

To illustrate the solution by determinants, let us solve the system of Equations (1) and (2) given in Example 1.

Example 3. Solve

$$2x - 3y = 13,$$
$$5x + 6y = -8.$$

Solution. *By Determinants:*

$$x = \frac{\begin{vmatrix} 13 & -3 \\ -8 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 5 & 6 \end{vmatrix}} = \frac{(13)(6) - (-8)(-3)}{(2)(6) - (5)(-3)}$$
$$= \frac{78 - 24}{12 + 15} = \frac{54}{27} = 2,$$

$$y = \frac{\begin{vmatrix} 2 & 13 \\ 5 & -8 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ 5 & 6 \end{vmatrix}} = \frac{(2)(-8) - (5)(13)}{27}$$

$$= \frac{-16 - 65}{27} = \frac{-81}{27} = -3.$$

If the graphs of the two linear equations do not intersect in a common point, the system of equations will not have a solution. This will be the case when the two lines either coincide or are parallel; that is, if the two lines have the same slope. The two lines will have the same slope if $a_1b_2 - a_2b_1 = 0$; that is, if the determinant of the system is equal to zero. For, if $a_1b_2 - a_2b_1 = 0$, then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} \quad \text{or} \quad -\frac{a_1}{b_1} = -\frac{a_2}{b_2}, \tag{12}$$

which are the respective slopes of the two lines. To show that $-\frac{a_1}{b_1}$ is the slope of the line $a_1x + b_1y = c_1$, we solve for y, as follows:

$$y = -\frac{a_1}{b_1}x + \frac{c_1}{b_1}.$$

Then, differentiating, we find that

$$\text{slope} = \frac{dy}{dx} = -\frac{a_1}{b_1}.$$

Likewise, $-\frac{a_2}{b_2}$ is the slope of the line $a_2x + b_2y = c_2$.

Consider the system of equations

$$a_1x + b_1y = c_1,$$
$$a_2x + b_2y = c_2.$$

I. *If the determinant of the system is not equal to zero; that is, if*

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0,$$

the equations are said to be **consistent** *and independent, the system has* **a solution,** *and the two lines* **intersect.**

II. If

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0,$$

and if either or both of the determinants in the numerators of the solution is nonzero, the equations are said to be ***inconsistent,*** *the system* ***does not have a solution,*** *and the two lines are* ***parallel.***

III. If

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0,$$

and if both of the determinants in the numerators of the solution are zero, the equations are said to be dependent, the system has ***infinitely many solutions,*** *and the two lines* ***coincide.***

Method of Graphs

Graph the two equations on the same coördinate system, then estimate the coördinates of the point of intersection of the two graphs. These coördinates are the solution.

Let us solve by graphs the following system of equations.

Example 4. Solve

$$2x - 3y = 13, \quad (1)$$
$$5x + 6y = -8. \quad (2)$$

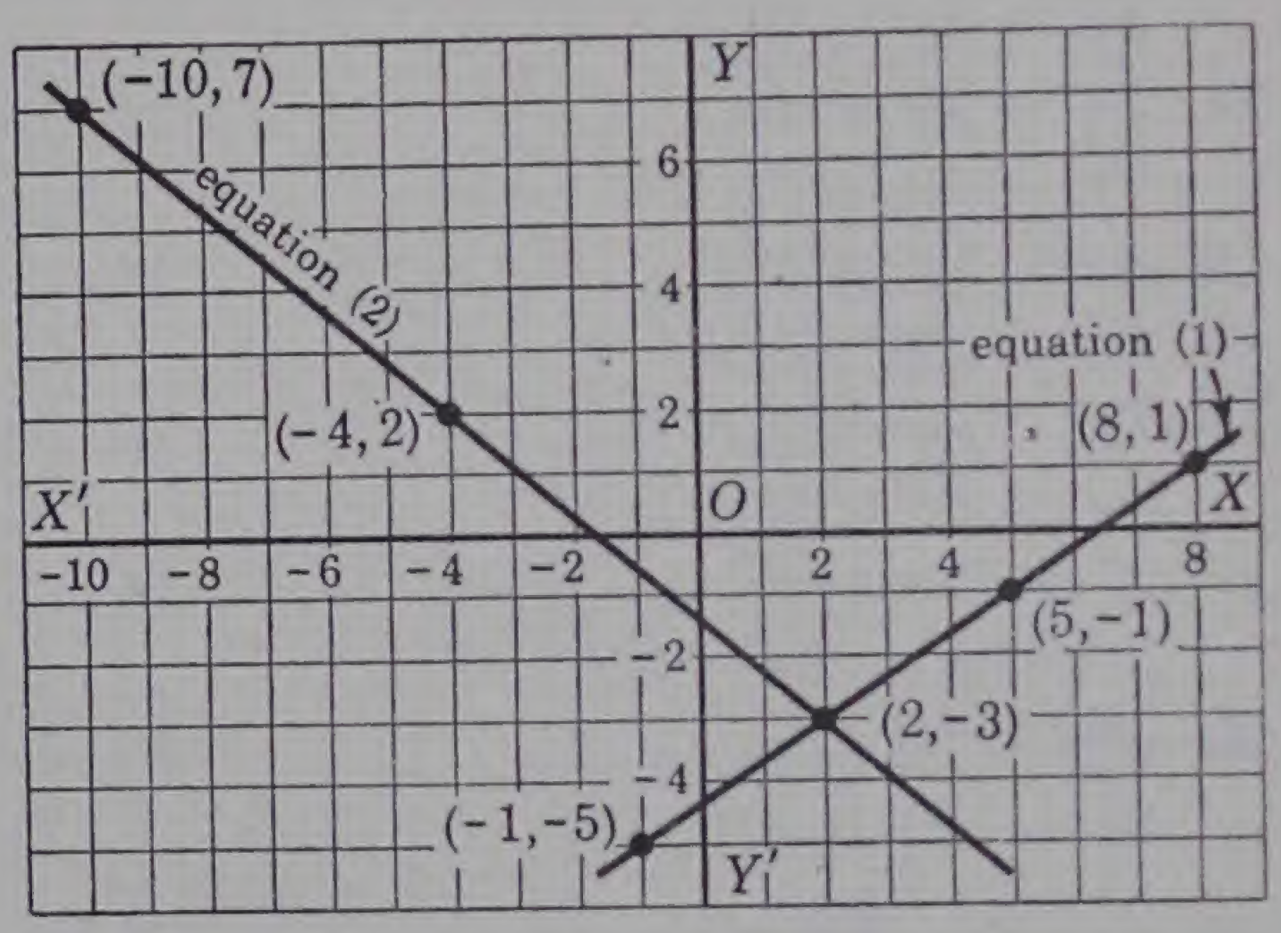


FIG. 16.

Graphical Solution. Since the graph of Equation (1) contains all the points whose coördinates satisfy Equation (1), and the graph of Equation (2) contains all the points whose coördinates satisfy Equation (2), it follows that the only pair of values of x and y that will satisfy each equation are the coördinates of the point of intersection of the two graphs.

The graph of each equation is a straight line and the graphs may be drawn from the accompanying sets of calculated data.

For Equation (1)

$$x = \frac{3y + 13}{2}$$

x	y
8	1
5	−1
−1	−5

For Equation (2)

$$x = \frac{-6y - 8}{5}$$

x	y
−4	2
−10	7
8	−8

From the graphs in Fig. 16 we see that the two lines intersect at the point (2, −3), which is the solution of the given system of equations.

EXERCISES

1. Graph and solve by elimination: $3x - y = 3$, $x + 2y = 8$.
2. Graph and solve by substitution: $3x + 2y = 4$, $4x + 3y = 7$.
3. Graph and solve by determinants: $3x + 8y = 8$, $6x - 4y = 1$.
4. Graph and solve by any method: $5x - 2y = 3$, $2x + 3y = -14$.

Solve each of the following systems of equations (Exercises 5 through 38) by any method and check the results.

5. $3x = y + 16$,
$3y = x - 24$.

6. $3x - 2y = -1$,
$6x - 5y = -7$.

7. $3x - 2y + 16 = 0$,
$2x + 5y - 21 = 0$.

8. $9x - 2y = 100$,
$6y - 5x = -80$.

9. $x + 2 = 0$,
$3x + 5y = 0$.

10. $x = 6y + 37$,
$y = 2x + 3$.

11. $3x + 2y = 10$,
$7x - 4y = 0$.

12. $y = 3x + 4$,
$y = \frac{2}{3}x - \frac{5}{3}$.

13. $6x - 8y = 10$,
$9x - 12y = 15$.

14. $3x - 2y = 7$,
$2x - 3y = 6$.

15. $2x - 5y + 7 = 0$,
$3x + 8y - 9 = 0$.

16. $6x = 3y - 5$,
$6y = 2x - 3$.

17. $2 - x - 4y = 0$,
$5 - 3x - 2y = 0$.

18. $8x - 15y = 73$,
$10x + 24y = -37$.

19. $x(y - 5) = y(x - 1)$,
$2(x - 2) = 3(y + 3)$.

20. $3x - \frac{1}{5}(2y + 3) = 2$,
$2y - \frac{3}{2}(x - 2) = 8$.

21. $\dfrac{x - 5}{5} = \dfrac{3y + 2}{10}$,

$\dfrac{x}{6} + 2 = \dfrac{y + 12}{4}$,

22. $3\sqrt{x} - 2\sqrt{y} = 11$,
$2\sqrt{x} + 5\sqrt{y} = 20$.

23. $2x - 1 = \dfrac{6y - 2}{3}$,

$3y - 1 = \dfrac{4x - 2}{5}$.

24. $\dfrac{x - y}{x + y} = \dfrac{2}{5}$,

$\dfrac{x - 10}{3y + 2} = \dfrac{2x - 3}{6y + 11}$.

25. $\dfrac{x - 1}{2} + \dfrac{y - 1}{3} = \dfrac{x + 1}{3} + \dfrac{y - 2}{6}$,
$(x - 3)(y - 2) = (x - 1)(y - 4)$.

26. $\dfrac{x - y + 1}{x - y - 1} = a$,

$\dfrac{x + y + 1}{x + y - 1} = b$.

27. $x + y = a$,
$x - y = b$.

28. $ax - 3y = 7$,
$4x - ay = 8$.

29. $ax + by = a^2 + b^2,$
$bx + ay = 2ab.$

30. $(a - b)x = (a + b)y,$
$ax = by + a^2 + b^2.$

31. $a(a - y) = b(x + y - a),$
$a(x - b - y) = b(x - b).$

32. $\frac{2}{3x} - \frac{3}{2y} - 5 = 0,$
$4y + 9x + 6xy = 0.$

33. $\frac{5x}{2} - \frac{3y}{5} = 6,$
$\frac{3y}{10} - \frac{15x}{4} = -7.$

34. $\frac{3x}{4} - \frac{2y}{3} + \frac{1}{2} = 0,$
$\frac{2x}{3} - \frac{y}{3} - \frac{3}{2} = 0.$

35. $\frac{4}{x} - \frac{1}{y} = 7,$
$\frac{7}{x} - \frac{5}{y} = -4.$

$\left(\textit{Hint:}\ \text{Solve for } \frac{1}{x} \text{ and } \frac{1}{y}\right).$

36. $\frac{1}{x} + \frac{1}{y} = 11,$
$\frac{3}{x} + \frac{4}{y} = 25.$

37. $\frac{2}{3x} - \frac{3}{4y} = 2,$
$\frac{4}{9x} + \frac{3}{5y} = -\frac{2}{15}.$

38. $\frac{1}{x} - \frac{2}{y} = -\frac{25}{6},$
$\frac{3}{x} - \frac{4}{y} = -\frac{59}{6}.$

39. In the formulas $l = a + (n - 1)d$, $s = \frac{n}{2}(a + l)$,

(a) Solve for a and s in terms of n, l, and d.
(b) Solve for a and l in terms of n, d, and s.
(c) Solve for s and n in terms of a, d, and l.
(d) Solve for a and d in terms of n, s, and l.
(e) Solve for l and d in terms of a, n, and s.
(f) Solve for s and d in terms of a, n, and l.

40. In the formulas $l = ar^{n-1}$, $s = \frac{a(r^n - 1)}{r - 1}$,

(a) Solve for a and l in terms of s, n, and r.
(b) Solve for a and s in terms of n, l, and r.
(c) Solve for r in terms of s, a, and l.

41. Two numbers are so related that their sum is 35 and three times the one number is twice the other number. Find the numbers.

42. If the length of a rectangle is decreased by 10 feet and the width is increased by 8 feet, the resulting figure is a square of the same area. Find the dimensions of the rectangle.

43. Two numbers are written with the same two digits. If the sum of the numbers is 110 and the difference of the numbers is 36, find the numbers.

44. The sum of the two digits of a number is 12, and if 18 is added to the number, the order of the digits will be reversed. Find the number.

45. A certain fraction becomes equal to 2 when 13 is added to the numerator and equal to 5 when 8 is subtracted from the denominator. Find the fraction.

46. A certain fraction becomes $\frac{2}{7}$ when 4 is added to the numerator and $\frac{5}{3}$ when 4 is subtracted from the denominator. Find the fraction.

47. A father is ten times as old as his son, and 4 years hence will be six times as old as the son then is. What are their ages?

48. A father is now four times as old as his son, but 4 years hence the father will be three times as old as the son. What are their present ages?

49. A rowing crew which must practice on a river wishes to determine their speed in still water. They observe that they can row 16 miles downstream in 1 hour, but it takes them 2 hours to row back. What is their rate in still water? What is the rate of the current?

50. A boat can go downstream a certain distance in 1 hour and 12 minutes but takes 3 hours to return. If the rate of the stream is 3 miles per hour, find the distance traveled and the rate of the boat in still water.

51. It is desired to make some green paint by mixing yellow and blue pigments. A gallon of paint was made by using 5 parts of yellow pigment and 2 parts of blue pigment, but this was found to be too light a green. On a second attempt a gallon of paint was made by using 7 parts yellow and 4 parts blue, but this was found to be too dark a green. It was decided that 2 parts yellow and 1 part blue would give (approximately) the desired shade. How much paint must be taken from each gallon to give a gallon of paint of the desired shade?

52. With two mixtures of a chemical and water of $a\%$ and $b\%$ concentrations respectively, how much from each mixture must be taken to give a mixture of q ounces of $c\%$ concentration?

53. The price of admission tickets to a certain theater is 50 cents for adults and 30 cents for children. If 686 tickets are sold and the total receipts are \$319, how many adults and how many children attended the theater?

54. The annual income from an investment of \$2,000 is \$76. If part of it is invested at 3% simple interest, and the remainder at 5%, what amount is invested at each rate?

55. A merchant bought 12 dresses and 10 coats for \$640. He sold the dresses at a profit of 40% and the coats at a profit of 30%, and received in all \$856. Find the cost of each dress and each coat.

56. A merchant paid \$2,400 for a quantity of dresses at \$20 each and of coats at \$50 each. He sold the dresses at a profit of 30% and the

coats at a profit of 20% and made in all a profit of \$570. How many dresses and how many coats did he buy?

57. How long will it be from the time the two hands of a clock are together until they will be together again?

58. A rowing crew which must practice on a river wishes to determine their speed in still water. They observe that they can row d miles downstream in t_1 hours, but it takes them t_2 hours to row back. Find their rate of rowing in still water and the rate of the current.

59. This exercise is the old problem, "How old is Ann?" The combined ages of Mary and Ann is 44. Mary is twice as old as Ann was, when Mary was half as old as Ann will be, when Ann is three times as old as Mary was, when Mary was three times as old as Ann.

60. A grocer has some coffee worth 42 cents per pound and some worth 28 cents per pound. What proportions of each must be used to give a mixture worth 34 cents per pound?

61. The general equation of a straight line is $y = mx + b$. (a) Find the equation of the straight line passing through the points $(2, -3)$ and $(-3, 4)$. (b) Through the points $(3, -2)$ and $(-3, -5)$. (*Hint:* Substitute the coördinates of each point in the equation $y = mx + b$ and solve for m and b.)

62. A basket contains between 5 and 6 dozen of eggs. A person counting the eggs 3 at a time finds there are 2 left over, but when he counts them 4 at a time he finds there are 3 left over. How many eggs are in the basket?

51. SYSTEMS OF LINEAR EQUATIONS IN THREE UNKNOWNS

The graph of an equation in two variables is in a plane, or space of two dimensions. The graph of an equation in three variables must be in a space of three dimensions. The graph of a linear equation in two variables is a straight line. The graph of a linear equation in three variables is a plane. Since in general two planes intersect in a line, and three planes intersect in a point, it follows that the solution for three unknowns in a system of linear equations is possible only if the system contains three independent equations in the three unknowns.

We shall give three methods for solving a system of three linear equations in three unknowns, as follows:

1. *Method of elimination by addition and subtraction.*
2. *Method of elimination by substitution.*
3. *Method of determinants.*

Method of Elimination by Addition and Subtraction

In a system of three linear equations in three unknowns, it is possible to eliminate one of the unknowns from any pair of equations by multiplying each term of the one equation by some constant and multiplying each term of the other equation by a different constant, so that this unknown will have the same coefficient in the two resulting equations. Then, if the members of these two equations are added or subtracted depending on the signs, we have an equation in but two unknowns. In the same manner it is possible to eliminate the *same unknown* from either of the other two pairs of equations. The resulting two equations in the two unknowns can now be solved for these two unknowns by any of the methods given in Article 50.

To illustrate, let us solve the following system of equations by addition and subtraction.

Example 1. Solve

$$x - 2y + 3z = 12, \tag{13}$$
$$3x + 4y - 2z = -7, \tag{14}$$
$$2x + 5y + 2z = 3. \tag{15}$$

Solution. Eliminate z from the pair of Equations (13) and (14) by multiplying each term of Equation (13) by 2 and each term of Equation (14) by 3 and then adding the members of the two resulting equations; thus:

$$\begin{array}{rrrr} 2x - & 4y + 6z = & 24 & \\ 9x + & 12y - 6z = & -21 & \\ \hline 11x + & 8y \qquad = & 3 & \end{array} \tag{16}$$

Eliminate z from the pair of Equations (14) and (15) by adding the members of the two equations; thus:

$$\begin{array}{rrrr} 3x + & 4y - 2z = & -7 & \\ 2x + & 5y + 2z = & 3 & \\ \hline 5x + & 9y \qquad = & -4 & \end{array} \tag{17}$$

Let us now solve the system composed of the two equations (16) and (17)

$$11x + 8y = 3, \tag{16}$$
$$5x + 9y = -4, \tag{17}$$

for x and y by addition and subtraction, as follows:

$$\begin{array}{rrr} 99x + 72y = & 27 \\ 40x + 72y = & -32 \\ \hline 59x = & 59 \\ x = & 1 \end{array} \qquad \begin{array}{rrr} 55x + 40y = & 15 \\ 55x + 99y = & -44 \\ \hline -59y = & 59 \\ y = & -1 \end{array}$$

To solve for z let us substitute these values of x and y in Equation (13). Doing this, we obtain

$$1 + 2 + 3z = 12;$$

solving for z gives

$$z = 3.$$

Therefore, the solution of the system of Equations (13), (14), and (15) by addition and subtraction is $x = 1$, $y = -1$, and $z = 3$. Sometimes this solution is written as $(1, -1, 3)$.

Check: To check, substitute these values of x, y, and z in Equations (14) and (15).

Substituting in (14) gives

$$3 - 4 - 6 = -7;$$

substituting in (15) gives

$$2 - 5 + 6 = 3.$$

Method of Elimination by Substitution

A system of three linear equations in three unknowns can be solved by solving any one of the three equations for one of the unknowns in terms of the other two unknowns, and substituting the result in each of the two remaining equations. The two resulting equations will contain but two unknowns, and this system of equations can be solved for these two unknowns by any of the methods given in Article 50.

Let us solve by substitution the system of equations given in Example 1.

Example 2. Solve

$$x - 2y + 3z = 12, \tag{13}$$
$$3x + 4y - 2z = -7, \tag{14}$$
$$2x + 5y + 2z = 3. \tag{15}$$

Solution. Suppose we solve Equation (13) for x in terms of y and z, and then substitute this value for x in Equations (14) and (15). Solving Equation (13) for x, we have

$$x = 12 + 2y - 3z. \tag{18}$$

Substituting for x in (14), we obtain

$$3(12 + 2y - 3z) + 4y - 2z = -7;$$

substituting for x in (15) gives

$$2(12 + 2y - 3z) + 5y + 2z = 3.$$

Simplifying, we have

$$10y - 11z = -43, \tag{19}$$

and

$$9y - 4z = -21. \tag{20}$$

Let us next solve the system of Equations (19) and (20) for y and z by substitution. To do this let us solve Equation (19) for y in terms of z, and then substitute the result in Equation (20). Solving Equation (19) for y, we have

$$y = \frac{11z - 43}{10}, \tag{21}$$

substituting for y in (20) gives

$$9\left(\frac{11z - 43}{10}\right) - 4z = -21;$$

simplifying, we get

$$99z - 387 - 40z = -210;$$

then

$$59z = 177,$$

and

$$z = 3.$$

We substitute this value for z in Equation (21), thus:

$$y = \frac{11z - 43}{10} = \frac{33 - 43}{10} = \frac{-10}{10} = -1,$$

and then substitute the values for y and z in Equation (18):

$$x = 12 + 2y - 3z = 12 - 2 - 9 = 1.$$

Hence, the solution of the system of Equations (13), (14), and (15) by substitution is $x = 1$, $y = -1$, and $z = 3$, sometimes written as $(1, -1, 3)$.

Check: To check, substitute these results in Equations (14) and (15). Substituting in (14) gives

$$3 - 4 - 6 = -7;$$

substituting in (15) gives

$$2 - 5 + 6 = 3.$$

Method of Determinants

Let us solve by the method of elimination the system of three general linear equations

$$a_1x + b_1y + c_1z = d_1, \tag{22}$$
$$a_2x + b_2y + c_2z = d_2, \tag{23}$$
$$a_3x + b_3y + c_3z = d_3, \tag{24}$$

and then express these results in determinant form.

It is possible to eliminate two of the unknowns at one time instead of eliminating one unknown at a time. We can eliminate y and z by multiplying the members of Equations (22), (23), and (24) by

$$(b_2c_3 - b_3c_2), \qquad -(b_1c_3 - b_3c_1), \qquad (b_1c_2 - b_2c_1),$$

respectively, and then adding the members of the resulting equations. In this sum, the coefficients of y and z are zero and the result is

$$\begin{aligned}&[a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)]x \\ &= d_1(b_2c_3 - b_3c_2) - d_2(b_1c_3 - b_3c_1) + d_3(b_1c_2 - b_2c_1). \qquad (25)\end{aligned}$$

Removing parentheses and solving for x, writing the positive products first, we have

$$x = \frac{d_1b_2c_3 + d_3b_1c_2 + d_2b_3c_1 - d_3b_2c_1 - d_1b_3c_2 - d_2b_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}. \qquad (26)$$

In a similar manner it is possible to solve for y by eliminating x and z by multiplying the members of Equations (22), (23), and (24) by

$$-(a_2c_3 - a_3c_2), \qquad (a_1c_3 - a_3c_1), \qquad -(a_1c_2 - a_2c_1),$$

respectively, and then adding the members of the resulting equations. Doing this we would get

$$y = \frac{a_1d_2c_3 + a_3d_1c_2 + a_2d_3c_1 - a_3d_2c_1 - a_1d_3c_2 - a_2d_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}. \qquad (27)$$

We can solve for z by eliminating x and y by multiplying the members of Equations (22), (23), and (24) by

$$(a_2b_3 - a_3b_2), \qquad -(a_1b_3 - a_3b_1), \qquad (a_1b_2 - a_2b_1),$$

respectively, and then adding the members of the resulting equations. Doing this we would get

$$z = \frac{a_1b_2d_3 + a_3b_1d_2 + a_2b_3d_1 - a_3b_2d_1 - a_1b_3d_2 - a_2b_1d_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}. \qquad (28)$$

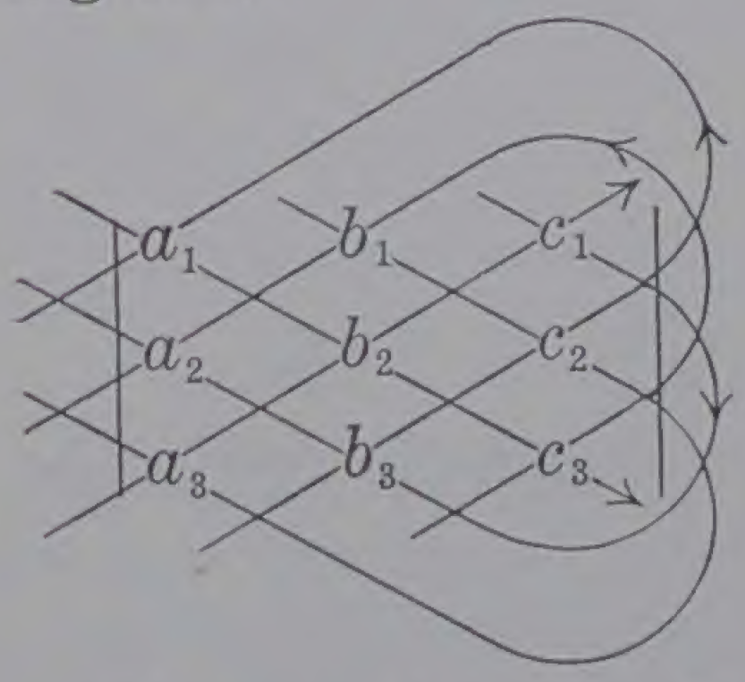
Since we wrote the denominators in the solution of a system of two linear equations in two unknowns in a symbol form as a square array of the coefficients of the unknowns, we shall also write the denominators in the solution of a system of three linear equations in three unknowns in a symbol form as a square array of the coefficients of the three unknowns. Thus we write

$$a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$$
$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \qquad (29)$$

The symbol on the right side of Equation (29) is called a **determinant of order three.** It is necessary for this symbol always to have the same meaning; that is, *it must be the algebraic sum of six products of three letters each.* Observe that each of the six products contains a letter from each row and from each column of the determinant and that the six products include all possible products that can be formed by selecting a letter from each row and each column (like letters denote columns and like subscripts denote rows in the determinant given in Equation 29).

The algebraic sum of the six products in Equation (29) is called the **expansion** of the determinant.

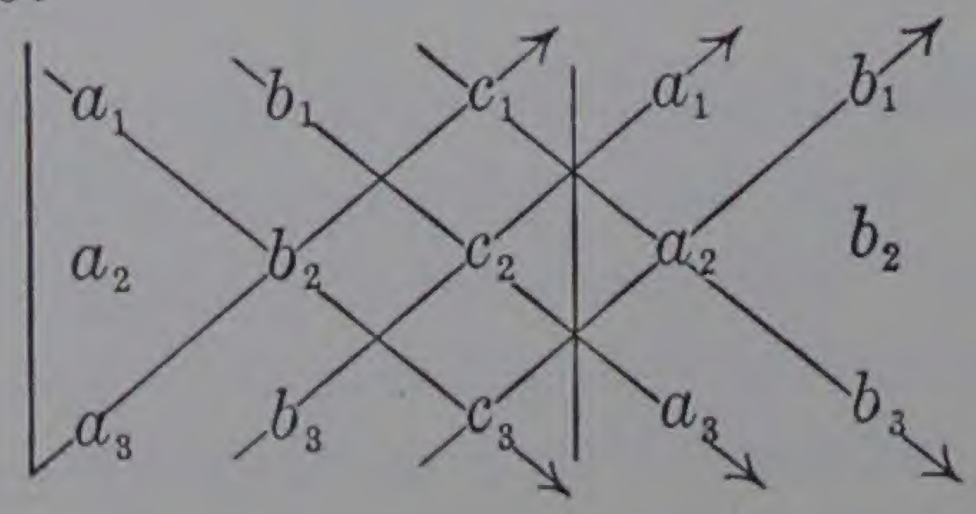
To be able to write the expansion of a determinant of order three, it is best to have a systematic way of forming the products. An easy way to obtain these products is to observe the systematic order in which the letters can be selected from the square array to form each product when the products are written in the following order



$$= a_1b_2c_3 + b_1c_2a_3 + c_1b_3a_2 - a_3b_2c_1 - b_3c_2a_1 - c_3b_1a_2. \qquad (30)$$

Observe that the first three products are preceded by a plus sign, and the last three products are preceded by a minus sign.

The expansion of a determinant of order three may also be obtained by rewriting the first and second columns to the right of the determinant as:



Then all the elements of any product lie on a diagonal. *The products of the elements in each of the three diagonals running down from left to right are given positive signs, and the products of the elements in each of the three diagonals running up from left to right are given negative signs. The algebraic sum of the six products thus formed is called the expansion of the determinant.*

To illustrate the expansion of a determinant of order three, let us expand the determinant composed of the coefficients of equations (13), (14), and (15). The determinant and its expansion is

$$\begin{vmatrix} 1 & -2 & 3 \\ 3 & 4 & -2 \\ 2 & 5 & 2 \end{vmatrix} = 1 \cdot 4 \cdot 2 + (-2)(-2)2 + 3 \cdot 5 \cdot 3 - 2 \cdot 4 \cdot 3 - 5(-2)1 - 2(-2)3$$
$$= 8 + 8 + 45 - 24 + 10 + 12$$
$$= 59.$$

The values of x, y, and z given in Equations (26), (27), and (28) for the solution of the following system of equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3, \end{aligned}\right\} \tag{31}$$

can be written in determinant form thus:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

$$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

provided the determinant in the denominator is not equal to zero. The determinant in the denominator is called the **determinant of the system** of equations.

Since Equations (31) constitute the solution of the three general linear equations (22), (23), and (24), we may consider them as the formula for the solution of any system of three linear equations in three unknowns, where proper values are substituted for the a's, b's, c's, and d's.

Observe that the three given equations are written with the terms containing the unknowns in the left-hand members and the constant terms in the right-hand members. The determinant of the system is a square array of the coefficients of the three unknowns, of which the first column consists of the coefficients of the one unknown, the second column consists of the coefficients of the second unknown, and the third column consists of the coefficients of the third unknown. The value of each unknown has the determinant of the system for its denominator and the determinant in the numerator is like the determinant in the denominator except that the column containing the coefficients of the unknown for which we are solving is replaced by the respective constants on the right-hand side of the equality signs.

To illustrate how to solve a system of three linear equations in three unknowns by the use of formulas given in Equation (31), let us solve the following system of equations by determinants.

Example 3. Solve

$$x - 2y + 3z = 12, \tag{13}$$
$$3x + 4y - 2z = -7, \tag{14}$$
$$2x + 5y + 2z = 3. \tag{15}$$

Solution. *By Determinants.*

$$x = \frac{\begin{vmatrix} 12 & -2 & 3 \\ -7 & 4 & -2 \\ 3 & 5 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & -2 & 3 \\ 3 & 4 & -2 \\ 2 & 5 & 2 \end{vmatrix}}$$

$$= \frac{96 + 12 - 105 - 36 + 120 - 28}{8 + 8 + 45 - 24 + 10 + 12} = \frac{59}{59} = 1,$$

$$y = \frac{\begin{vmatrix} 1 & 12 & 3 \\ 3 & -7 & -2 \\ 2 & 3 & 2 \end{vmatrix}}{59}$$

$$= \frac{-14 - 48 + 27 + 42 + 6 - 72}{59} = \frac{-59}{59} = -1,$$

$$z = \frac{\begin{vmatrix} 1 & -2 & 12 \\ 3 & 4 & -7 \\ 2 & 5 & 3 \end{vmatrix}}{59}$$

$$= \frac{12 + 28 + 180 - 96 + 35 + 18}{59} = \frac{177}{59} = 3.$$

Check: To check, substitute these results in any one of the three equations. Substituting in Equation (13), we have

$$1 + 2 + 9 = 12.$$

If the determinant of the system is equal to zero; that is, if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0,$$

then the three equations are either inconsistent or dependent, and the graphs of the three equations will have one of the following relations. (See *First Course in the Theory of Equations*, by Dickson, page 117.)

(a) Two of the planes intersect in a line parallel to the third plane.
(b) The three planes are parallel and not all of them are coincident.
(c) Two of the planes are parallel and cut by the third plane.
(d) The three planes intersect in a common line.
(e) The three planes coincide.
(f) Two of the planes coincide and are cut by the third plane.

For the three cases (a), (b), and (c), the three equations are **inconsistent,** and the system does not have a solution. For the three cases (d), (e), and (f), the three equations are **dependent,** and the system has an unlimited number of solutions.

QUESTIONS

1. What is meant by a solution of a system of equations?

2. What must be the relation between the number of equations and the number of unknowns in a system, if the system has a finite number of solutions?

3. Give four methods for solving a system of two linear equations in two unknowns.

4. Give three methods for solving a system of three linear equations in three unknowns.

EXERCISES

Solve each of the following systems of equations by addition and subtraction, by substitution, and by determinants.

1. $x + y + z = 3;\ x - 2y + 3z = 1;\ 3x + 5y - 4z = -1.$
2. $2x + 3y + 4z = 1;\ 3x - y - 5z = -4;\ 5x + 2y - 3z = -7.$
3. $3x - 4y + 2z = 0;\ 5x + y - 3z = -1;\ 8x - 2y - 4z = -12.$

Solve each of the systems of equations in Exercises 4 through 24 by any method. Check the results.

4. $3x + 2y - 4z = 7,$
$2x - 3y - 2z = 12,$
$x + y - 3z = 5.$

5. $x + y = -7,$
$y + z = -2,$
$z + x = 1.$

6. $x + y - z = 0,$
$x - 2y + z = 0,$
$4x - 5y + 2z = 0.$

7. $x - 3y - 2z = 3,$
$2y + 3z = 8,$
$4z - 3x = 1.$

8. $x = 2y - 3z + 4,$
$y = 2z - 3x + 1,$
$z = 2x - 3y - 1.$

9. $3x + 2y + z + 1 = 0,$
$3y - 6z + 11 - 4x = 0,$
$2z - 5x - 4y = 0.$

10. $3x - 2y = 12,$
$3y - 2z = -19,$
$3z - 2x = 11.$

11. $3x = 2y - 4z + 4,$
$4y - 8z = 2 - 9x,$
$5 + 6x - 6y - 8z = 0.$

12. $x - 2y = 7,$
$2y - 3z = -5,$
$3z - 4x = 4.$

13. $2x - 3y - z = -11,$
$13 + 4x - 2y = 3z,$
$2z + 3 - 3x = 5y.$

14. $2x - 4y + 6z = -\frac{37}{6},$
$4x - 6y + 2z = -\frac{7}{2},$
$6x - 2y + 4z = -\frac{4}{3}.$

15. $ax + by + cz = a,$
$bx + cy + az = b,$
$cx + ay + bz = c.$

16. $bx - 2y - az = -a^2,$
$ax + y - az = 2ab,$
$x - z = b.$

17. $\frac{1}{x} - \frac{3}{y} + \frac{2}{z} - 4 = 0,$
$\frac{5}{x} + \frac{6}{y} + \frac{4}{z} - 11 = 0,$
$\frac{2}{x} + \frac{8}{y} - \frac{3}{z} - 8 = 0.$

18. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 9,$
$\frac{2}{x} - \frac{3}{y} - \frac{1}{z} = -9,$
$\frac{3}{x} - \frac{2}{y} + \frac{4}{z} = 16.$

19. $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = -1,$
$\frac{3x}{2} - \frac{y}{3} - \frac{5z}{2} = 6,$
$\frac{5x}{2} + \frac{8y}{3} + \frac{3z}{2} = -3.$

20. $\frac{2}{3x} + \frac{3}{4y} + \frac{4}{5z} = 1,$
$\frac{4}{9x} - \frac{5}{8y} + \frac{14}{5z} = 5,$
$\frac{3}{2x} + \frac{9}{2y} - \frac{5}{z} = -10.$

21. $\frac{x}{y-1} = -\frac{1}{3},$
$\frac{y}{z-2} = -2,$
$\frac{z}{x-3} = -\frac{3}{2}.$

22. $3x - y + 2z - 7 = 0$; $6 + 3y - 2x - 7z = 0$;
$3z + 1 + 4x + 2y = 0$.

23. $3x + 4y + 5z - 3 = 0$; $4 + 9x - 8y + 10z = 0$;
$15z - 16y - 6x + 22 = 0$.

24. $4x + 8z = 9y + 6$; $3x + 10z = 1 + 6y$; $15x + 12y = 28 + 18z$.

25. Three numbers have the property that their sums in pairs are 16, 21, and 25, respectively. Find the numbers.

26. In a triangle the first angle is twice a second angle, the second angle is three times the third angle. Find the three angles.

27. A and B together can earn \$82.50 in 10 days; A and C together can earn \$58.50 in 6 days; B and C together can earn \$108.00 in 12 days. What does each earn per day?

28. A sum of money is divided among 3 men so that the first man gets \$10 less than one-half of all the money, the second man gets \$5 more than one-half of the remaining money, and the third man gets one-third as much as the first two. How much does each man get?

29. Three numbers have the property that if their average be added successively to each of the numbers, the resulting sums are 10, 18, and 20. Find the numbers.

30. A certain number is expressed by three digits whose sum is 12. If the tens' and hundreds' digits be interchanged the number is increased by 90. One-half of the sum of the units' and tens' digits equals the sum of the tens' and hundreds' digits. Find the number.

31. Three circles, with centers at the vertices of a triangle whose sides are 15, 20, and 25 inches long, are drawn tangent in pairs. Find the diameters of each of the circles.

32. The general equation of a circle is $x^2 + y^2 + ax + by + c = 0$. (a) Find the equation of the circle passing through the three points $(5, 1)$, $(-2, 0)$, and $(-1, -7)$. (b) Through the three points $(1, -1)$, $(7, -9)$, and $(8, -2)$.

33. A man buys 100 pets for \$195. He pays \$20 apiece for the dogs; \$5 apiece for the cats; and \$1 apiece for the rabbits. How many of each did he buy?

34. A collection of nickels, dimes, and quarters totals \$5.50. The total number of coins is 58, and there are three times as many nickels as dimes. How many of each kind are there?

35. In making a journey of 350 miles through rolling country the driver observed that he could average 40 miles per hour on the level, 30 miles per hour uphill, and 45 miles per hour downhill. If it took $9\frac{1}{3}$ hours to go and 9 hours to return, how many miles of the road was level, how many miles uphill, how many miles downhill?

CHAPTER IX

Ratio, Proportion, and Variation

52. RATIO AND PROPORTION

The **ratio** *of a number* a *to a number* b $(b \neq 0)$ *is the quotient* $\frac{a}{b}$.

The ratio $\frac{a}{b}$ is sometimes denoted by $a:b$, which is read "a is to b."

The ratio is used to express the relative magnitude of two quantities; the quantities compared must be expressed in the same unit.

Illustration: The ratio of 75 cents to \$3.00 is $75:300 = \frac{75}{300} = \frac{1}{4}$.

If we say distance A and distance B are in the ratio 2:3, we mean distance A is $\frac{2}{3}$ of distance B. In this case no mention is made of the magnitude of either distance A or distance B.

A ***proportion*** *is a statement that two ratios are equal.* The four numbers a, b, c, and d are said to be in **proportion** if $a/b = c/d$. This proportion can also be written in the form $a:b = c:d$, or sometimes as $a:b::c:d$. This proportion is read "a is to b as c is to d," where a **and** d are called the **extremes,** b **and** c are called the **means,** and d is called the **fourth proportional** to a, b, and c.

If $a:x = x:b$, then x is called a **mean proportional** between a and b.

By means of the operations of algebra it is possible to transform the proportion $a:b = c:d$ into many different forms. The different forms can be readily obtained from the equation $\frac{a}{b} = \frac{c}{d}$.

THEOREM I. *If* $a:b = c:d$, *then* $ad = bc$. *In any proportion the product of the means equals the product of the extremes.*

THEOREM II. *If* $a:b = c:d$, *then* $a:c = b:d$. *In any proportion the means may be interchanged.*

THEOREM III. *If* $a:b = c:d$, *then* $b:a = d:c$. The second proportion is said to be obtained from the first by **inversion.**

THEOREM IV. *If* $a:x = x:b$, *then* $x^2 = ab$ *or* $x = \pm\sqrt{ab}$. *The mean proportion between* a *and* b *is* $\pm\sqrt{ab}$.

THEOREM V. *If* $a:b = c:d$, *then* $\dfrac{a+b}{b} = \dfrac{c+d}{d}$. (*Hint for proof:* Simplify $\dfrac{a}{b} + 1 = \dfrac{c}{d} + 1$.)

THEOREM VI. *If* $a:b = c:d$, *then* $\dfrac{a-b}{b} = \dfrac{c-d}{d}$. (*Hint for proof:* Simplify $\dfrac{a}{b} - 1 = \dfrac{c}{d} - 1$.)

THEOREM VII. *If* $a:b = c:d$, *then* $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$. This can be obtained by using the results of Theorem V and Theorem VI.

THEOREM VIII. *If* $a:b:c:d = A:B:C:D$, *then* $\dfrac{a}{A} = \dfrac{b}{B} = \dfrac{c}{C} = \dfrac{d}{D}$ and $\dfrac{a+b+c+d}{A+B+C+D} = \dfrac{a}{A}$.

Example. A father divides \$3,000 among his three sons in the ratio 4:5:6. How much did each son receive?

Solution. Since the ratios of the parts are 4:5:6, the parts themselves may be taken to be $4k$, $5k$, $6k$, provided

$$4k + 5k + 6k = 3{,}000.$$

Hence,

$$15k = 3{,}000,$$

and

$$k = 200.$$

Therefore the amounts are

\$800, \$1,000, \$1,200.

EXERCISES

Express each ratio in Exercises 1 through 12 as a fraction in simplest form, or as a decimal.

1. 12 feet to 15 feet.
2. 5 miles to 2.5 miles.
3. 3 feet to 18 inches.
4. 32 inches to 4 feet.
5. 3 days to 36 hours.
6. 40 hours to 5 days.
7. 44 feet per second to 60 miles per hour.
8. 720 miles per hour to 792 feet per second.
9. 25 pounds per square inch to 2,880 pounds per square foot.

10. The length of a side of a square to that of its diagonal.
11. The circumference of a circle to its diameter.
12. The area of a circle to the square of the radius.

Find the fourth proportional to each of the three given quantities in Exercises 13 through 20.

13. 2, 3, 8. **14.** 17, 2, 34. **15.** 3, 5, 9. **16.** 1, 2, 3.
17. a, b, a^2. **18.** a, ab, b. **19.** a, b, ab. **20.** a, a^2, b.

Find the mean proportional between each of the pairs of numbers in Exercises 21 through 28.

21. 3, 12. **22.** 2, 8. **23.** 4, 16. **24.** 1, 25.
25. $\frac{1}{3}$, $\frac{1}{12}$. **26.** $\frac{2}{3}$, $\frac{3}{8}$. **27.** 8, $\frac{1}{32}$. **28.** $\frac{1}{a}$, a^3.

29. A line segment 28 inches long is divided into two parts whose lengths have the ratio 3:4. Find the length of each part.
30. A line segment 150 inches long is divided into two parts whose lengths have the ratio 2:3. Find the length of each part.
31. A line segment 2 feet long is divided into two parts whose lengths have the ratio 1:2. Find the length of each part.
32. A line segment 36 inches long is divided into three parts whose lengths have the ratio 2:3:4. Find the length of each part.
33. A line segment 5 inches long is divided into three parts whose lengths are in the ratio 3:5:7. Find the length of each part.
34. Divide \$4,500 among four men in the ratio 1:2:3:4. How much does each man receive?
35. The three sides of a triangle are 6, 9, and 12 inches, respectively. What are the sides of a similar triangle if the shortest side is 4 inches long?
36. The three sides of a triangle are 8, 10, and 15 inches, respectively. What are the sides of a similar triangle if the longest side is 20 inches?
37. If a train travels at uniform speed and travels 105 miles in 4 hours, how far will the train travel in 6 hours?
38. If an auto travels at uniform speed and travels 210 miles in 4 hours, how long will it take the auto to travel 350 miles?
39. A man 5 feet 8 inches tall casts a shadow 10 feet 5 inches long. At the same time of day, a building casts a shadow 250 feet long. How high is the building?

53. VARIATION

It may happen that the manner in which one quantity varies with a second quantity is known, but the exact relation or function connecting the two quantities is not known. For example, we know that distance in uniform motion varies with time, but we

may not know the velocity. However, if a single observation is made and it is found that the distance is 10 miles at the end of 2 hours, then we know that the velocity is 5 miles per hour, and thus the relation or function connecting distance and time in this case is completely determined. The calculation may be carried out as follows. Let S represent distance in miles and T time in hours; then

$$S = f(T) = V_0 T.$$

For $S = 10$ and $T = 2$, we have

$$10 = V_0 2,$$

solving for V_0, we get

$$V_0 = 5.$$

Hence, the functional relation connecting S and T in this example is

$$S = f(T) = 5T.$$

When we say that one quantity *varies* as a second quantity, or is *proportional* to a second quantity, we mean that the *ratio of the two quantities is constant;* that is, the one quantity is equal to a constant times the other quantity. To illustrate, if y varies as x, then we may write

$$\frac{y}{x} = k \qquad \text{or} \qquad y = kx. \tag{1}$$

If y varies as x^2, then

$$\frac{y}{x^2} = k \qquad \text{or} \qquad y = kx^2. \tag{2}$$

If one quantity *varies inversely* as a second quantity, it varies as the *reciprocal* of this quantity. If y varies inversely as x^3, then

$$y = \frac{k}{x^3}. \tag{3}$$

If y varies as u and v and inversely as w^2, then

$$y = k\,\frac{u \cdot v}{w^2}. \tag{4}$$

In all the exercises to follow, the functional relation connecting the variables is known except for a constant that is called the **constant of variation,** or **constant of proportionality.** However, a single observation, or set of data, is sufficient to determine this constant,

and hence will determine completely the functional relation connecting the variables.

It is also possible in many problems of this kind to set up a connecting relation between the known quantities and the unknown quantities by using proportions. When the relation is set up by proportion, it is not necessary to use a constant of variation.

Example 1. In uniform motion the distance S varies with the time T. If the distance is 16 miles at the end of 40 minutes, find the distance S at the end of 2 hours.

It is necessary to use the same unit of measure for a given letter in both sets of data. It helps to keep the units straight if we record the given data in tabular form as shown in the accompanying table (expressing time in minutes and distance in miles). In this form the same units must be used throughout a given column. The tabular form will also aid in substituting the correct numbers for the right letters.

S	T
16 miles	40 minutes
?	120 minutes

Solution. *By Method of Variation.* Let

$$S = kT.$$

If $S = 16$ and $T = 40$; then

$$16 = 40k.$$

Solving for k gives

$$k = \frac{2}{5}.$$

Hence, the formula for distance is

$$S = \frac{2}{5}T.$$

For $T = 120$,

$$S = \frac{2}{5}(120) = 48 \text{ miles.}$$

Solution. *By Proportion.* Since distances are proportional to time

$$S:16 = 120:40,$$

or

$$\frac{S}{16} = \frac{120}{40} = 3,$$

hence,

$$S = 3 \cdot 16 = 48 \text{ miles.}$$

This is nothing more than the ratio of the members of the two equations $S = 120k$ and $16 = 40k$, obtained by substituting the two sets of data

in the equation $S = kT$. Thus, by taking the ratio of corresponding members of the two equations, we have

$$\frac{S}{16} = \frac{120k}{40k} = 3.$$

Solving for S gives

$$S = 3 \cdot 16 = 48 \text{ miles.}$$

In general, the method of variation is easier to use than the method of proportion. This is especially true if the problem contains more than two variables.

If we wish to solve Example 1 by expressing time in hours, we have

$$S = kT,$$

hence,

$$16 = \frac{2}{3}k.$$

Solving for k gives

$$k = \frac{3 \cdot 16}{2} = 24;$$

therefore, our formula is

$$S = 24T.$$

For $T = 2$,

$$S = 24 \cdot 2 = 48 \text{ miles.}$$

S	T
16 miles	$\frac{2}{3}$ hours
?	2 hours

Example 2. The cost C to paint the surface of a sphere varies as the square of the radius r and the number of coats n. If it costs \$4.20 to paint a sphere of 3 feet radius with 7 coats of paint, how much will it cost to paint a sphere of 5 feet radius with 3 coats of paint?

Solution. In this case the functional relation connecting the letters is

$$C = kr^2n.$$

Substituting, we have

$$420 = k3^2 \cdot 7.$$

Solving for k gives

$$k = \frac{420}{3^2 \cdot 7} = \frac{20}{3};$$

therefore, our formula is

$$C = \tfrac{20}{3} r^2n.$$

For $r = 5$ and $n = 3$,

$$C = \tfrac{20}{3} 5^2 \cdot 3 = 500 \quad \text{or} \quad \$5.00.$$

C	r	n
420 cents	3 feet	7
?	5 feet	3

It is possible to solve for C by taking the ratio of the members of the two equations $C = k5^2 \cdot 3$ and $420 = k3^2 \cdot 7$. Thus, we have

$$\frac{C}{420} = \frac{k5^2 \cdot 3}{k3^2 \cdot 7} = \frac{25}{21}.$$

Solving for C, we get

$$C = \tfrac{25}{21}(420) = 500 \qquad \text{or} \qquad \$5.00.$$

In Example 2, $k = \dfrac{20}{3}$ was calculated on the condition that the cost is given in cents and the radius in feet. Thus, the formula $C = \dfrac{20}{3} r^2 n$ gives the cost in cents of painting a sphere of radius r feet with n coats of paint, provided the cost of paint is the same as in the given data.

Problems in variation can be solved either by computing the value of the constant of variation and thus determining completely the formula connecting the different variables, or by eliminating the constant of variation without first finding its value. The latter is accomplished by taking the ratio of the members of the two equations formed by substituting the two given sets of data in the equation connecting these variables.

QUESTIONS

1. What is meant by saying that one quantity varies as a second quantity?
2. What is meant by saying that one quantity varies inversely as a second quantity?
3. What is meant by the constant of variation?
4. How may problems in variation be solved?

EXERCISES

1. State in words the following fifteen equations, in which k denotes the constant of variation.

(a) $y = kx$; (b) $R = kv^2$; (c) $E = kRI$;

(d) $z = kxy$; (e) $V = kr^3$; (f) $A = P(1 + R)^n$;

(g) $z = k\dfrac{x^2}{y}$; (h) $L = k\dfrac{bd^2}{l}$; (i) $D = k\dfrac{M - m}{M + m}$;

(j) $w = k\dfrac{x^2 y}{z}$; (k) $d = k\dfrac{Wl^3}{bd^3}$; (l) $t = k\sqrt{\dfrac{l}{g}}$;

(m) $w = k\dfrac{x^2\sqrt{y}}{z^3}$; (n) $s = k\dfrac{l}{r^2}$; (o) $s = kgt^2$.

Express each of the following (Exercises 2 through 9) in equation form.

2. (a) y varies as x; (b) z varies as x and inversely as y; (c) w varies as x^2 and y^2 and inversely as z^3; (d) z varies as the sum of the squares of x and y and inversely as the difference of their squares; (e) y varies as x^2, and $y = 21$ when $x = 7$;

3. The volume of a cylinder varies as the height and the square of its radius.

4. The area of a triangle varies as the base and the altitude.

5. The force exerted by the wind on a sail varies as the area of the sail and the square of the wind's velocity.

6. The value of an automobile varies inversely with its age.

7. The resistance of a wire to an electric current varies directly as its length and inversely as the area of its cross section.

8. The time of a complete vibration of a pendulum varies as the square root of its length.

9. The number of vibrations of a pendulum varies inversely as the square root of its length.

10. Give the functional relation connecting the radii and number of revolutions of the front and rear wheels of a carriage.

11. If y varies as x and if $y = 12$ when $x = 5$, find y when $x = 10$.

12. If z varies as x and y and if $z = 5$ when $x = 7$ and $y = 2$, find z when $x = 2$ and $y = 7$.

13. If z varies as x and inversely as y and if $z = 7$ when $x = 5$ and $y = 4$, find z when $x = 10$ and $y = 3$.

14. If y varies as x^2 and is equal to 50 when $x = 5$, what is the relation between y and x?

15. If z varies as x and inversely as y and is equal to 10 when $x = 25$ and $y = 10$, what is the relation connecting x, y, and z?

16. Give the formula for finding the value of z, if z varies as x^2 and inversely as y^3, and $z = 18$ when $x = 3$ and $y = 2$.

17. The volume of a sphere varies as the cube of its radius, and the volume of a sphere of radius 2 feet is 33.5104 cubic feet. Give the formula for the volume of a sphere.

18. The volume of a cone varies as the altitude and the square of the radius of the base, and the volume of a cone of radius 5 feet and altitude 4 feet is 104.72 cubic feet. Give the formula for the volume of a cone.

19. The brake horsepower (B.H.P.) of an engine varies as the square of the diameter in inches of the cylinders and the number of cylinders. The B.H.P. of an eight-cylinder engine with cylinders $3\frac{1}{2}$ inches in diameter is 39.2. Give the formula for the B.H.P. of this engine.

20. If the gasoline consumed by an automobile varies with the distance and the square root of the velocity, and if the automobile uses 5 gallons of gasoline in traveling 82 miles at a speed of 40 miles per hour, find the

formula for distance traveled in terms of the velocity and gasoline consumed.

21. The time of a single vibration of a pendulum varies as the square root of the length of the pendulum. If a pendulum 99 centimeters long beats seconds, how long must a pendulum be to swing 4 times as fast?

22. The time of a single vibration of a pendulum varies as the square root of its length and inversely as the square root of the gravity constant, and the length of a 2-seconds pendulum, where gravity equals 980 [cm. per sec. per sec.] is 397.18 centimeters. Give the formula for finding the length of a pendulum in terms of the time for a single vibration.

23. The time required to complete a given amount of work varies inversely as the number of men working. If 3 men can do a piece of work in 24 days, how long will it take 8 men to do the same work?

24. If an automobile traveling at a uniform rate travels 80 miles in 2 hours and 30 minutes, how long will it take the automobile to travel 184 miles?

25. The maximum range of a projectile varies as the square of the initial velocity. If the range is 20,000 feet when the initial velocity is 800 feet per second, find the range when the initial velocity is 1,000 feet per second; 500 feet per second.

26. If the wages of 5 men for 4 weeks is $560, how many weeks will 7 men work for $980?

27. The safe load of a horizontal beam supported at both ends varies as its breadth and the square of its depth, and inversely as its length between supports. If a 2- by 6-inch white pine joist 10 feet long holds 800 pounds when placed edgewise, what is its safe load when placed flatwise? What is the safe load of a beam of the same material, 14 feet long, 4 inches wide, 1 foot thick, if placed edgewise?

28. The safe load of a horizontal beam supported at both ends varies as its breadth and the square of its depth, and inversely as its length between supports. If a plank 8 inches wide, 2 inches thick, and 12 feet long holds 500 pounds when placed flatwise, how many tons will a 8- by 8-inch beam of the same material 16 feet long support?

29. The volume of a gas varies inversely as the pressure when the temperature is constant. If a certain quantity of gas occupies 100 cubic feet when the pressure is 18 pounds per square foot, what pressure is necessary to reduce the volume to 65 cubic feet? To 45 cubic feet?

30. The gas law states that for a perfect gas the product of pressure and volume is proportional to the absolute temperature. If the pressure is 35 pounds per square inch when the volume is 180 cubic inches and the temperature is 72° C, what must be the volume at 400° C if the pressure is to be 60 pounds per square inch? (Zero absolute temperature is −273° C.)

31. The distance a body falls from rest varies as the square of the time traveled. If a body falls from rest 402 feet in 5 seconds, how far will it fall in 8 seconds? In 12 seconds?

32. The material for a certain closed box costs \$4.50. What is the cost of a similar box whose volume is $\frac{8}{27}$ as large, if the cost varies with the surface area?

33. The deflection at the center of a horizontal beam, supported at the two ends, with a weight hung at the center, varies as the weight and the cube of its length and inversely as its width and the cube of its depth. If a plank 12 feet long, 10 inches wide, and 2 inches thick is bent 3 inches by a weight of 170 pounds, how much will a beam of the same material be bent by a weight of 500 pounds if the beam is 8 feet long, 1 foot wide, and 4 inches thick?

34. The amount of stretch in a wire varies as the length of the wire and the stretching force and inversely as the square of the radius of the wire. If a copper wire 72 centimeters long and having a radius of 0.036 centimeter is stretched 0.144 centimeter by a weight of 10,000 grams, how much will a copper wire of the same radius and 140 centimeters long be stretched by a weight of 7,000 grams?

35. The intensity of light varies inversely as the square of the distance from the source of light. How far from the light must an object be placed to receive 5 times as much light as an object placed 80 feet away?

36. The intensity of illumination varies as the intensity of the source of light and inversely as the square of the distance from the source of light. If two lights are 30 feet apart and one of them is $\frac{2}{5}$ as bright as the other, at what point between them is the illumination from the two the same?

37. If one can see easily to read when a light is 5 feet away, where must a light twice as bright be placed to give the same illumination? (See Exercise 36.)

38. The lift force on a monoplane wing varies as the area of the wing and the square of the velocity. If a plane with 500 square feet can lift $1\frac{1}{2}$ tons when flying at 200 miles per hour, how many pounds can a similar plane with a wing area of 800 square feet lift when flying at 220 feet per second?

39. The weight of a body varies inversely as the square of its distance from the center of the earth. How far above the surface of the earth must a 175-pound person be to weigh 112 pounds? (The radius of the earth is 3,960 miles.)

40. The weight of a body at the surface of a planet varies as the mass of the planet and inversely as the square of the radius of the planet. How much would a man weighing 175 pounds on the earth weigh on the moon if the radius of the earth is 3,960 miles, the radius of the moon is

1,080 miles, and the mass of the moon is $\frac{1}{81.5}$ times the mass of the earth? How much would he weigh on Jupiter if the radius of Jupiter is 44,000 miles and its mass is 1,355 times the mass of the earth? How much would he weigh on Mars if the radius of Mars is 2,100 miles and its mass is $\frac{1}{9}$ the mass of the earth? How much would he weigh on the sun if the radius of the sun is 433,250 miles and its mass is 333,000 times the mass of the earth?

41. The squares of the periods of revolution of the planets about the sun are proportional to the cubes of their distances from the sun. It takes the planet Jupiter 4,332 days to revolve around the sun. What is the relation between the distance Jupiter is from the sun and the earth's distance from the sun?

42. The time required for a planet to revolve around the sun varies as the $\frac{3}{2}$ power of its distance from the sun. The planet Mars requires 687 days to revolve around the sun. How far away from the sun is Mars in terms of the earth's distance from the sun?

43. Two pulleys are connected by a belt. If the one pulley is 10 inches in diameter and makes 500 revolutions per minute, how many revolutions will the other pulley make if it is 15 inches in diameter? Prove that the number of revolutions the pulleys make are inversely proportional to their diameters.

44. If a heavier weight draws up a lighter weight by means of a cord passing over a pulley, then the number of feet passed over by each weight in a given time varies as the difference of the weights and inversely as their sum. If 9 pounds draw 7 pounds up 12 feet in 3 seconds, how high will 15 pounds draw 12 pounds in the same time?

45. The force necessary to hold a weight W on an inclined plane is given by the proportion $F:W = h:l$, where l is the length of the incline and h is the height the one end is above the other end. Find the force necessary to keep a barrel weighing 350 pounds under control when unloading it from a truck by permitting it to roll down a plank 10 feet long, if the truck is 4 feet high.

46. If the lift of a monoplane varies as the square of its velocity and the area of its wing, what additional load can be carried by doubling the velocity? If the area of the wing of a second plane is 80% of the area of the wing of the first plane, the velocity of the second plane must be what per cent greater than the velocity of the first plane, if the two planes are to have the same lifting power?

CHAPTER X

Complex Numbers

$$i = i = i^5 = \cdots = i^{4k+1}.$$
$$-1 = i^2 = i^6 = \cdots = i^{4k+2}.$$
$$-i = i^3 = i^7 = \cdots = i^{4k+3}.$$
$$1 = i^4 = i^8 = \cdots = i^{4k}.$$
(k = any integer)

54. COMPLEX NUMBERS

All numbers of the form $a + bi$ are called complex numbers, where a and b are any real numbers (rational or irrational) and $i^2 = -1$. The term free of i, as a, is called the real part of the complex number, and the term containing i, as bi, is called the

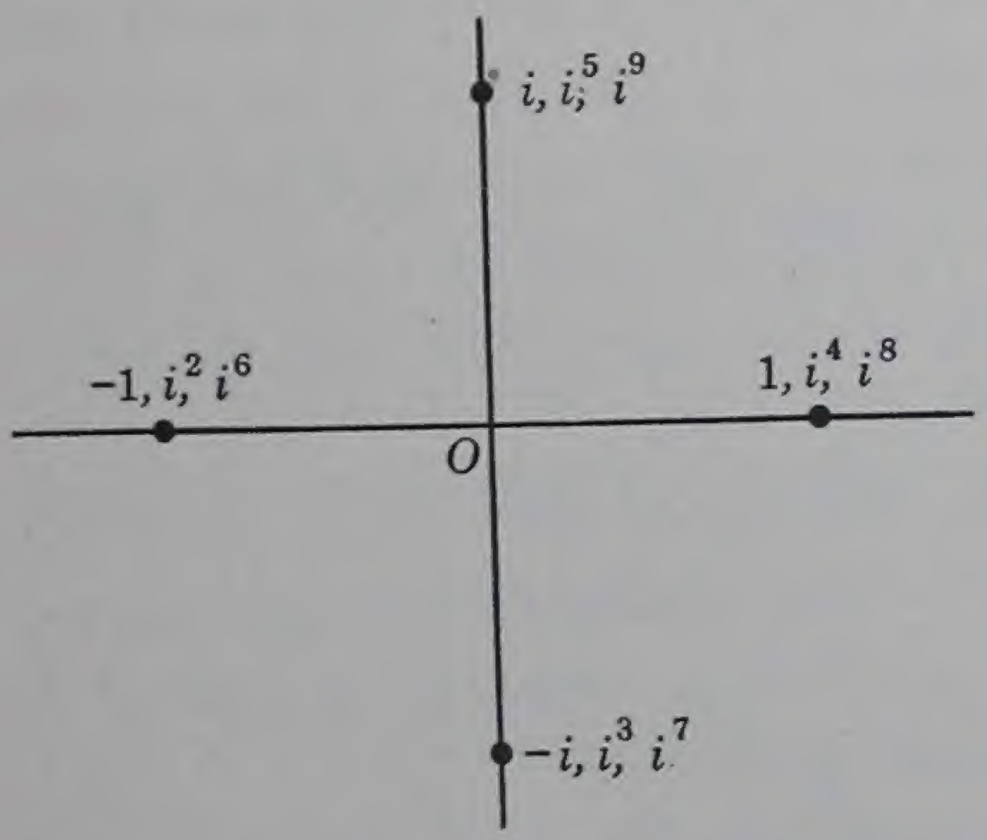


FIG. 17.

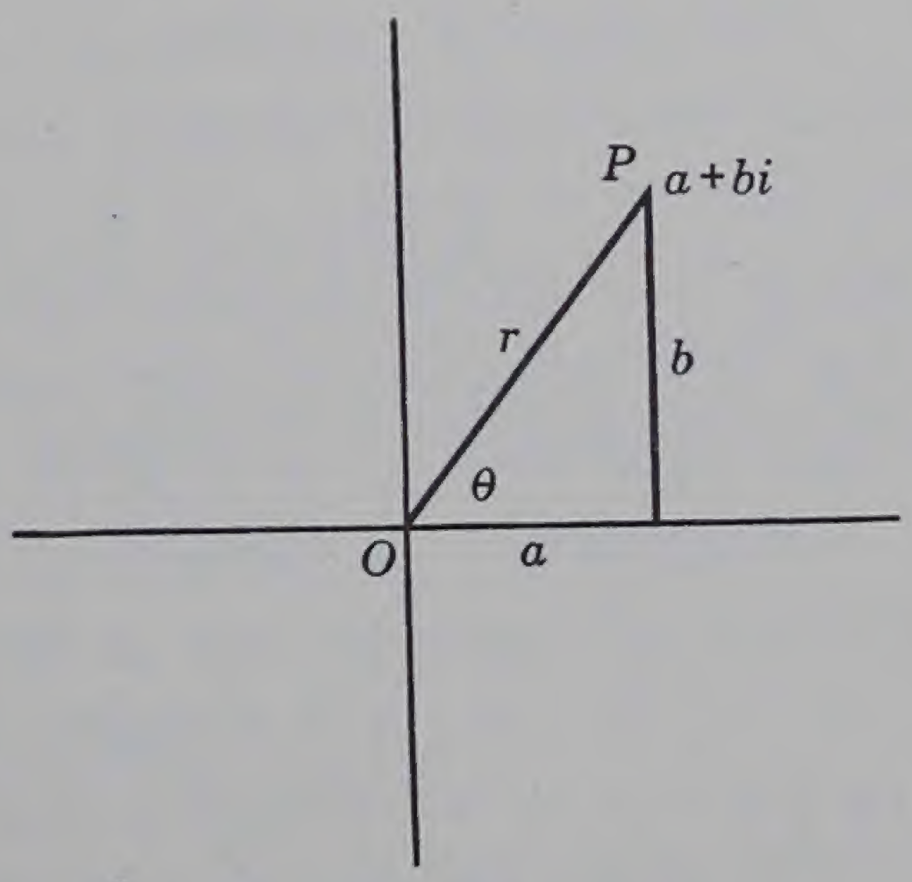


FIG. 18.

imaginary part of the complex number. The two complex numbers $a + bi$ and $a - bi$, which differ only in the signs of their imaginary parts, are called **conjugate complex numbers.**

Since $i = \sqrt{-1}$ and $i^2 = -1$, then $i^3 = i^2 \cdot i = -i$, $i^4 = (i^2)^2 = (-1)^2 = 1$, $i^5 = i^4 \cdot i = i$, and so on, wherefore we have the different powers of i as listed above.

By means of Article 9, the successive powers of i can be represented graphically in the complex plane as is illustrated in Fig. 17.

Since by Fig. 18, $a = r \cos \theta^*$ and $b = r \sin \theta$, it follows that there is a second way of representing the complex number $a + bi$ corresponding to the point P. Thus,

$$a + bi \quad \text{and} \quad r(\cos \theta + i \sin \theta)$$

are two ways of writing the same complex number, where $r = \sqrt{a^2 + b^2}$ is called the **modulus**, and θ is called the **amplitude** ($\tan \theta = a/b$).

$a + bi$ is called the *rectangular form.*

$r(\cos \theta + i \sin \theta)$ is called the **polar** or **trigonometric form.**

The following are examples of complex numbers written in each of the two forms.

$$16 = 16 + 0i = 16(\cos 0° + i \sin 0°). \quad (1)$$
$$-16 = -16 + 0i = 16(\cos 180° + i \sin 180°). \quad (2)$$
$$8i = 0 + 8i = 8(\cos 90° + i \sin 90°). \quad (3)$$
$$1 + i = \sqrt{2}\,(\cos 45° + i \sin 45°). \quad (4)$$

Two complex numbers are defined to be equal if their real parts are equal and their imaginary parts are equal. Thus, if $x + yi = a + bi$, then $x = a$ and $y = b$. Since zero written as a complex number is $0 + 0i$, it follows that *if a complex number is equal to zero, then the real part equals zero and the imaginary part equals zero.* That is, if $x + yi = 0$, then $x = 0$ and $y = 0$.

All expressions involving complex numbers combined a finite number of times by one or more of the five fundamental operations ($+, -, \times, \div, \sqrt[n]{}$) should be simplified to a complex number of the form $a + bi$, where a and b are real numbers.

It is possible to add, subtract, multiply, divide, and take powers of complex numbers either **algebraically** or **geometrically.**

Complex numbers are added, subtracted, and multiplied algebraically very much as algebraic expressions are added, subtracted, and multiplied.

* Those students who have not had trigonometry may experience some difficulty following the development of the trigonometric forms, but these students should at least observe the conclusions. The definitions of the trigonometric functions are given in Article 58.

55. ADDITION AND SUBTRACTION OF COMPLEX NUMBERS

To add complex numbers algebraically, take the algebraic sum of the real parts for the real part of the sum, and the algebraic sum of the coefficients of i for the coefficient of i in the sum. To illustrate:

Example. Find the sum $(2 + 3i) - (3 - 2i) + (-5 - 7i)$.

Solution.

$$(2 + 3i) - (3 - 2i) + (-5 - 7i) = (2 - 3 - 5) + (3 + 2 - 7)i$$
$$= -6 - 2i.$$

For the general case,

$$(a + bi) + (c + di) = (a + c) + (b + d)i. \qquad (5)$$

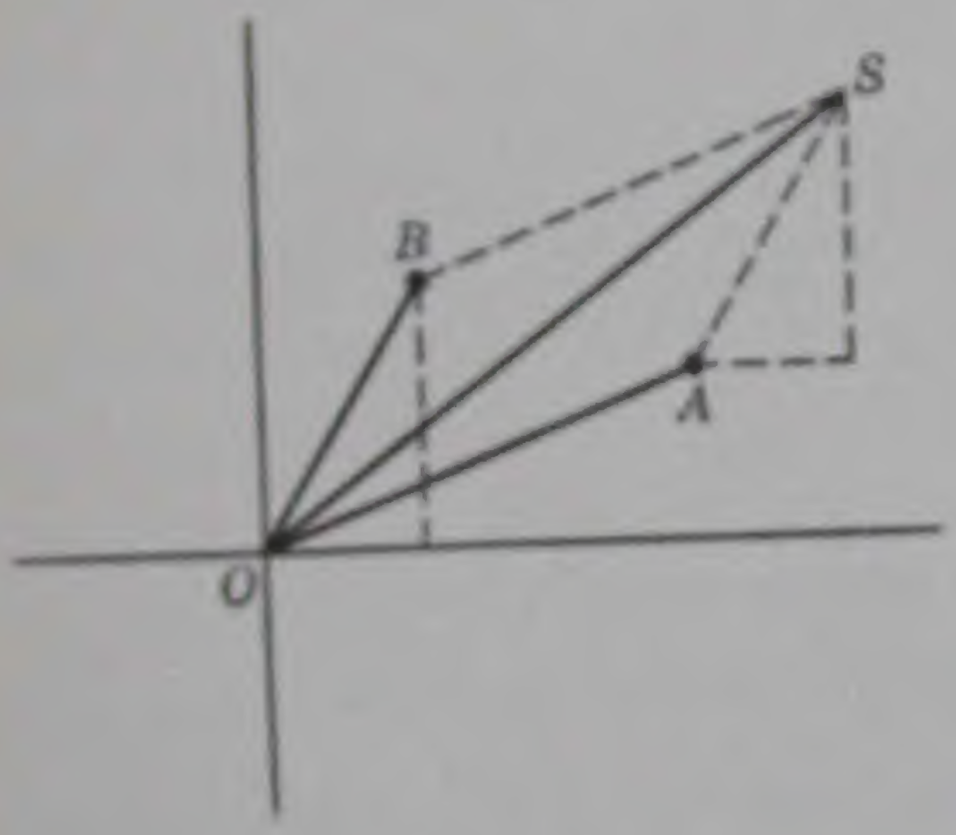


Fig. 19.

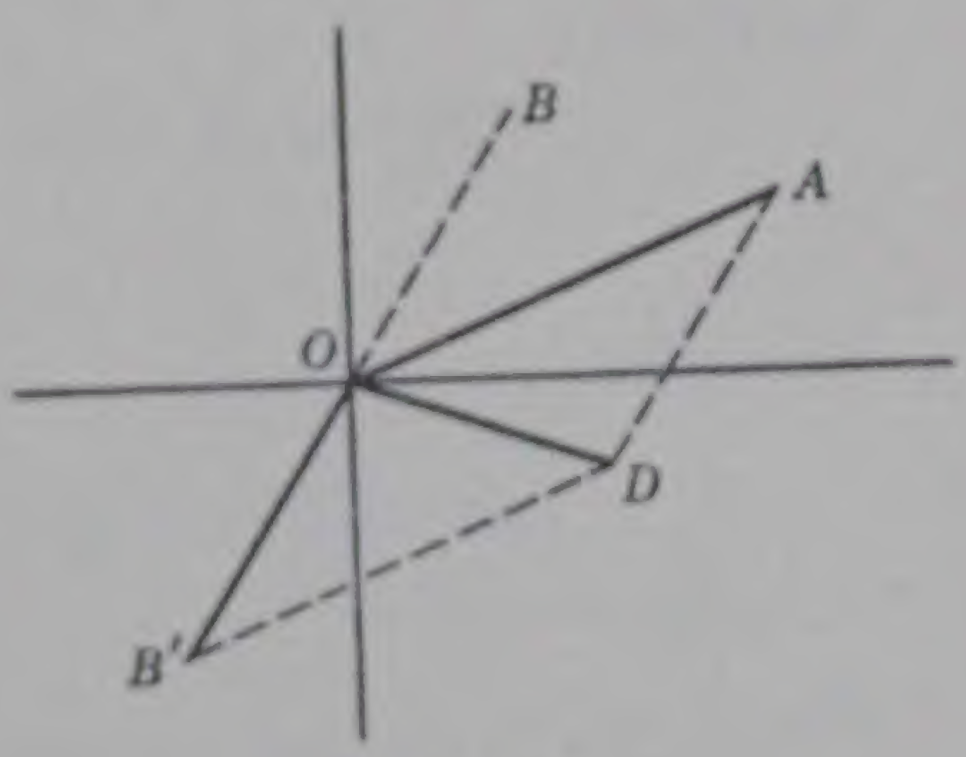


Fig. 20.

To add two complex numbers geometrically, let the point A in Fig. 19 represent the complex number $a + bi$, and the point B the number $c + di$, then the complex number corresponding to the point S, the vertex of the parallelogram $AOBS$, is the sum

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

This is evident from the construction in Fig. 19.

The difference $(a + bi) - (c + di)$ of two complex numbers can be found geometrically by finding the sum $(a + bi) + (-c - di)$ geometrically. Thus, to find the difference $(a + bi) - (c + di)$, plot the number $a + bi$, point A in Fig. 20, then, instead of plotting the number $c + di$, plot the number $-c - di$, point B'. The difference $(a + bi) - (c + di)$ is then represented by the point D, found by completing the parallelogram $AOB'D$ in Fig. 20. This difference is

$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

The point D representing the difference $(a + bi) - (c + di)$ can be located in another manner. For if we let $A = a + bi$ and $B = c + di$, and let $D = A - B$, then solving for A, $A = B + D$. Thus, with OB as one side of the parallelogram, construct the other side OD so that OA will be the diagonal.

56. MULTIPLICATION OF COMPLEX NUMBERS

To multiply two complex numbers, multiply each part of the one complex number respectively by each part of the other complex number, then after reducing i^2, *add the real parts for the real part in the product and add the coefficients of* i *for the coefficient of* i *in the product.* To illustrate:

Example 1. Find the product $(2 + 3i)(3 - 2i)$.

Solution.

$$\begin{aligned}(2 + 3i)(3 - 2i) &= 2(3 - 2i) + 3i(3 - 2i)\\ &= 6 - 4i + 9i - 6i^2\\ &= 12 + 5i.\end{aligned}$$

For the general product,

$$\begin{aligned}(a + bi)(c + di) &= ac + bci + adi + bdi^2\\ &= (ac - bd) + (ad + bc)i. \qquad (6)\end{aligned}$$

When multiplying several complex numbers together, either multiply all the numbers together and then reduce the higher powers of i, or multiply two numbers together and reduce i^2; then, multiply this result by the third complex number and reduce as before. To illustrate:

Example 2. Find the product $(2 - 3i)^2(3 + 4i)$.

Solution. *First Method.*

$$\begin{aligned}(2 - 3i)^2(3 + 4i) &= (4 - 12i + 9i^2)(3 + 4i)\\ &= 12 - 20i - 21i^2 + 36i^3\\ &= 33 - 56i.\end{aligned}$$

Solution. *Second Method.*

$$\begin{aligned}(2 - 3i)^2(3 + 4i) &= (-5 - 12i)(3 + 4i)\\ &= -15 - 56i - 48i^2\\ &= 33 - 56i,\end{aligned}$$

since $(2 - 3i)^2 = 4 - 12i + 9i^2 = -5 - 12i$.

From the product of two complex numbers in polar form, two very important observations can be made. Let us find the product of the two complex numbers

$$a + bi = r_1(\cos \theta_1 + i \sin \theta_1)$$

and

$$c + di = r_2(\cos \theta_2 + i \sin \theta_2);$$

multiplying, we have

$$\begin{aligned}(a + bi)(c + di) &= r_1 r_2[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + \\ &\qquad i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)] \\ &= r_1 r_2[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)]. \qquad (7)\end{aligned}$$

It follows from this that: (1) *The modulus of the product of two complex numbers is the product of the two moduli, and* (2) *the amplitude of the product is the sum of the two amplitudes.*

To find the product of two complex numbers $a + bi$ and $c + di$ geometrically, plot the two numbers as A and B, in Fig. 21; then construct the triangle OBP similar to triangle ODA, where the angle BOP equals θ_1 and OD is one unit long. The complex number corresponding to the point P is the product $(a + bi)(c + di)$.

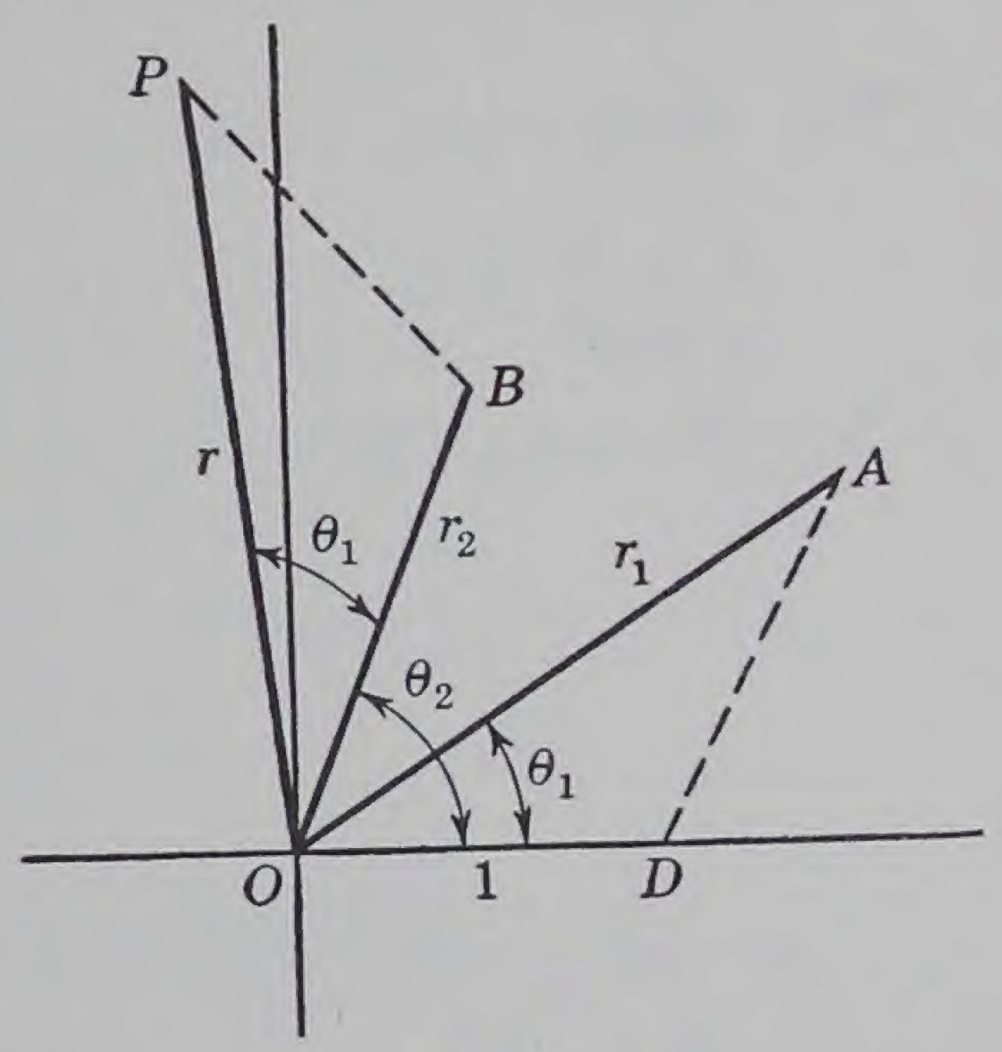


FIG. 21.

Proof: By construction the amplitude of OP is $(\theta_1 + \theta_2)$, and the modulus of OP is $r_1 r_2$ since, by similar triangles,

$$\frac{r}{r_2} = \frac{r_1}{1} \qquad \text{or} \qquad r = r_1 r_2. \qquad (8)$$

57. DIVISION OF COMPLEX NUMBERS

Since the quotient of two complex numbers, if the divisor is not zero, must be expressed in the form $a + bi$, the i in the denominator must be eliminated. This can be done similarly to rationalizing the denominator, since the product of any complex number and its conjugate is a real number. This product is

$$(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2. \qquad (9)$$

To simplify the quotient of two complex numbers: first, represent them in fraction form; then multiply both numerator and denominator

by the conjugate of the denominator. To illustrate:

Example. Simplify $(2 + 5i) \div (3 - 2i)$.

Solution. Writing this in fraction form and multiplying both numerator and denominator by $3 + 2i$, the conjugate of $3 - 2i$, we have

$$\frac{2 + 5i}{3 - 2i} = \frac{2 + 5i}{3 - 2i} \cdot \frac{3 + 2i}{3 + 2i} = \frac{(6 - 10) + (4 + 15)i}{9 + 4} = -\frac{4}{13} + \frac{19}{13}\,i.$$

For the general case

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{(ac + bd)}{c^2 + d^2} + \frac{(bc - ad)}{c^2 + d^2}\,i. \quad (10)$$

The ratio of two complex numbers in polar form is

$$\begin{aligned}\frac{a + bi}{c + di} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\ &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)} \\ &= \frac{r_1}{r_2}\left\{\frac{\left[\begin{array}{c}(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + \\ i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\end{array}\right]}{\cos^2\theta_2 + \sin^2\theta_2}\right\} \\ &= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]^* \quad (11)\end{aligned}$$

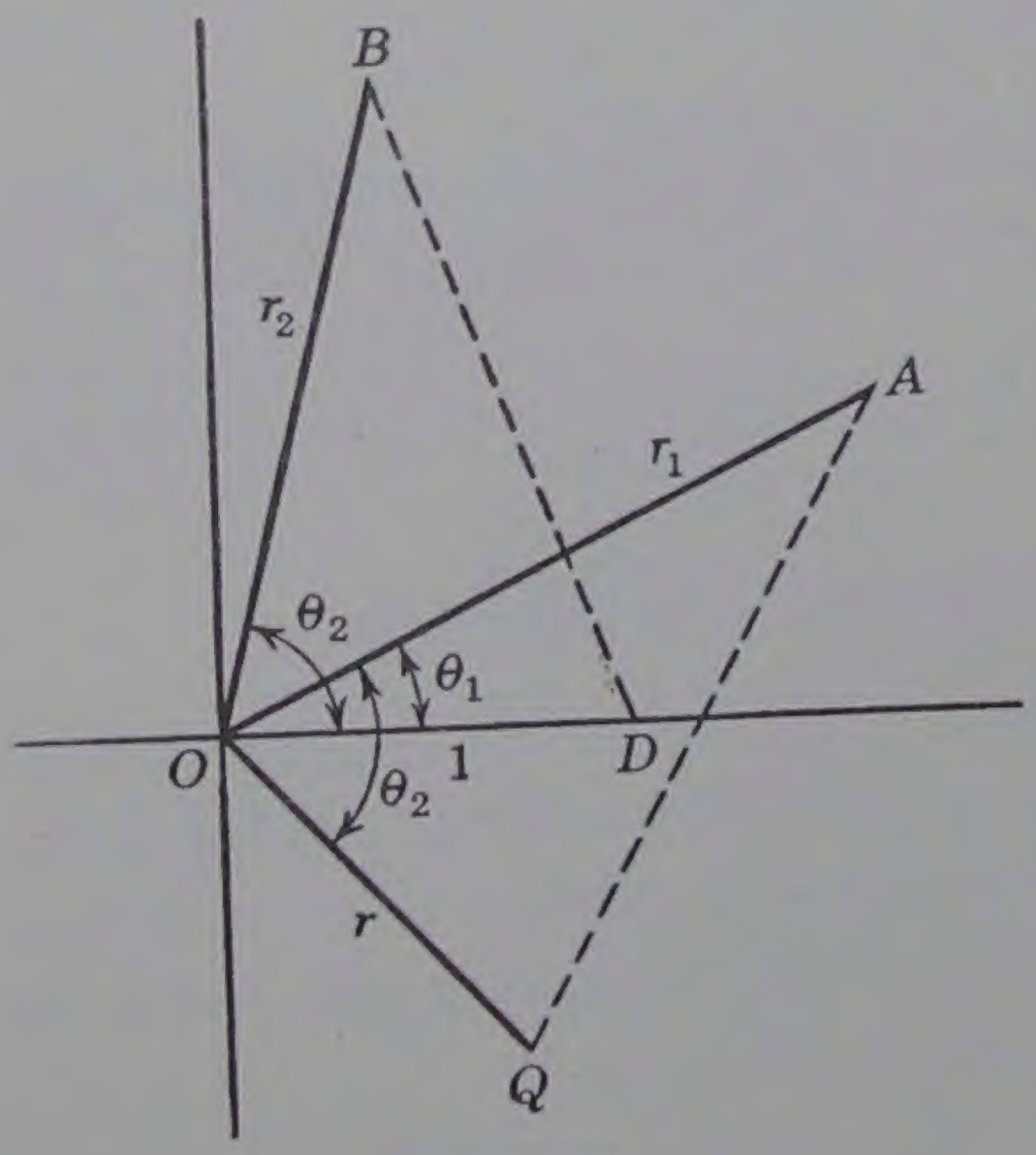


FIG. 22.

It follows from this that: (1) *The modulus of the quotient of two complex numbers is the modulus of the dividend divided by the modulus of the divisor and* (2) *the amplitude of the quotient is the difference of the two amplitudes* (*amplitude of the dividend minus the amplitude of the divisor*).

To find the quotient of two complex numbers, $a + bi$ and $c + di$, geometrically, plot the two numbers as A and B in Fig. 22, then construct the triangle OAQ similar to the triangle OBD,

* This result is obtained by using the following formulas found in trigonometry: $\sin^2 A + \cos^2 A = 1$; $\sin(A - B) = \sin A\cos B - \cos A\sin B$; $\cos(A - B) = \cos A\cos B + \sin A\sin B$.

where the angle AOQ equals θ_2 and OD is unity. The complex number corresponding to the point Q represents the quotient $(a + bi) \div (c + di)$.

Proof: By construction the amplitude of OQ is $\theta_1 - \theta_2$, and the modulus of OQ is $\frac{r_1}{r_2}$ since, by similar triangles,

$$\frac{r_2}{1} = \frac{r_1}{r} \qquad \text{or} \qquad r = \frac{r_1}{r_2}. \tag{12}$$

QUESTIONS

1. Give the meaning of i, and the value of its successive powers.
2. How are complex numbers added or subtracted?
3. How are complex numbers multiplied?
4. How are complex numbers divided?

EXERCISES

1. Write the following nine numbers in the form of a complex number as $a + bi$.

(a) $3 - 2\sqrt{-1}$; (b) $5 + \sqrt{-4}$; (c) $-7 - \sqrt{-3}$;
(d) $\sqrt{-24}$; (e) 8; (f) $\sqrt{7}$;
(g) 1; (h) 0; (i) i.

2. Write the following eight numbers in the form of $a + bi$.

(a) $2 - \sqrt{-3}$; (b) $3 + \sqrt{-4}$; (c) $\sqrt{5}$; (d) -8;
(e) $\sqrt{-16}$; (f) -1; (g) $-i$; (h) $2 - \sqrt{-2}$.

Perform the operations indicated in Exercises 3 through 38 and simplify each result.

3. $(2 + 3\sqrt{-1}) + (-5 + 6\sqrt{-1}) - (3 - 2\sqrt{1})$.

4. $(3 + 2i) - (6 - 2i) - (-5 + 6i)$.

5. $\sqrt{-3}\sqrt{-12}$. **6.** $\sqrt{-12} \div \sqrt{-3}$. **7.** $(2 - i)(3 - 2i)$.

8. $(3 - 2i)^2$. **9.** i^7. **10.** i^{-20}.

11. i^{35}. **12.** i^{95}. **13.** $(2i)^3$.

14. $(3i)^4$. **15.** $\frac{1}{i^3}$. **16.** $2i^{-5}$.

17. $\frac{3}{2i^9}$. **18.** $\frac{4}{(-i)^5}$. **19.** $3i^3 - 4i^2 - 2i + 1$.

20. $4i^5 - 5i^4 + 3i^2 - 2i - 6$.

21. $(3 + 2i) - (5 - 6i) - (-3 + 5i)$.

22. $2(3 - 4i) - 4(-3 + 6i) + (-4 - 7i)$.

23. $(2 - 3i)^3$.

24. $3(2 - 5i)(-3 + i)(2 - 3i)$.
25. $(3 - 2i)(1 + i)(2 + 3i)$.
26. $(3 - i)^2(1 - 3i)$.
27. $(2 - 3i)^2 - (-2 + 3i)^2$.
28. $(2 + 3i\sqrt{2})^2 - (2 - 3i\sqrt{2})^2$.

29. $\dfrac{2 - i}{3 + 2i}$. **30.** $\dfrac{1}{2 - 3i}$. **31.** $\dfrac{-2 + 3i}{3 - 4i}$.

32. $\dfrac{3 - 4i}{-4 - 3i}$. **33.** $\dfrac{(2 - 3i)^2}{-3 - 4i}$. **34.** $\dfrac{4 - 3i}{(-3 + 2i)^2}$.

35. $\left(\dfrac{1 - i\sqrt{2}}{1 + i\sqrt{3}}\right)^2$. **36.** $\left(\dfrac{3 + i\sqrt{2}}{3 - i\sqrt{2}}\right)^2$. **37.** $\dfrac{i + i^5}{i - i^5}$.

38. $\dfrac{(2 - i)(-3 - 2i)}{(3 - 2i)(-2 + i)}$.

39. Substitute i for x in $x^3 + 2x^2 + x + 2 = 0$ and simplify.
40. Substitute $1 - i$ for x in $x^2 - 2x + 2 = 0$ and simplify.
41. Substitute $-1 + 2i$ for x in $x^3 - x^2 - x - 15 = 0$ and simplify.
42. Is $1 - 2i$ a root of $x^3 + x^2 - x + 15 = 0$?
43. Is $1 + 2i$ a root of $x^3 - 5x^2 + 11x - 15 = 0$?
44. Is $2 - 3i$ a root of $x^3 - 2x^2 + 36 = 0$?
45. Is $-3 - 2i$ a root of $2x^3 + 9x^2 + 8x - 39 = 0$?
46. Is $-2 - i$ a root of $x^3 - 2x^2 - 3x + 7 = 0$?

58. TRIGONOMETRIC FUNCTIONS AND THEIR APPLICATIONS TO COMPLEX NUMBERS

The angle $XOP = \theta$ (see Fig. 23) is said to have the **initial side** ***OX***, the **terminal side** ***OP***, and the **vertex** ***O***. The angle θ (theta) is called a **positive angle,** if measured in a **counterclockwise** direction from the initial side OX, and is called a **negative angle,** if measured in a **clockwise** direction from OX.

In Fig. 23, let the coördinates of the point P be (x, y), let θ be the angle XOP, and let r be the distance OP which is always considered as positive. We shall now define in terms of x, y, and r the six trigonometric functions of θ as follows:

$$\sin\theta = \frac{y}{r} \quad \text{(read sine } \theta\text{)};$$

$$\cos\theta = \frac{x}{r} \quad \text{(read cosine } \theta\text{)};$$

$$\tan\theta = \frac{y}{x} \quad \text{(read tangent } \theta\text{)};$$

$$\csc\theta = \frac{r}{y} \quad \text{(read cosecant } \theta\text{)};$$

$$\sec\theta = \frac{r}{x} \quad \text{(read secant } \theta\text{)};$$

$$\cot\theta = \frac{x}{y} \quad \text{(read cotangent } \theta\text{)}.$$

Observe that for a given angle θ the value of each of these ratios, and hence the value of each of the trigonometric functions, remains

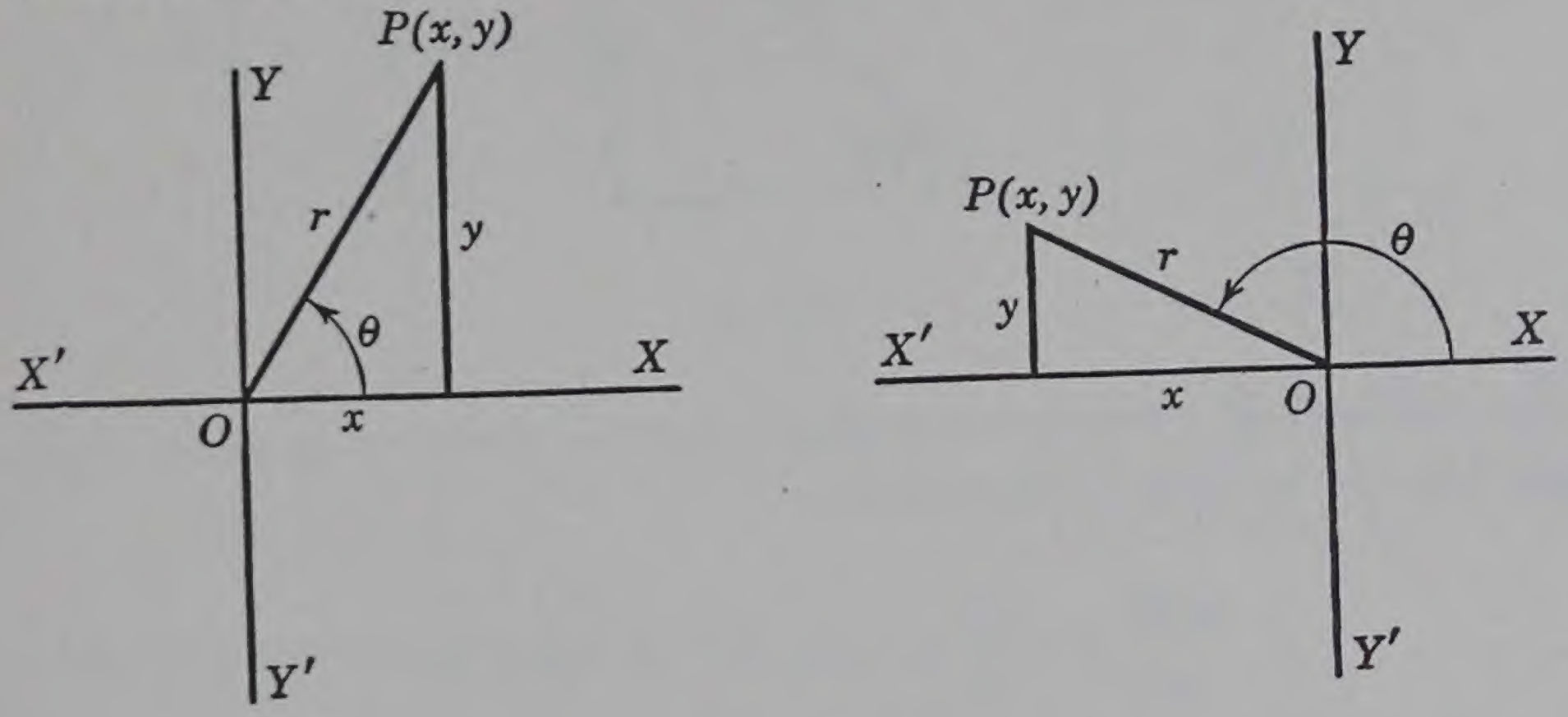


FIG. 23.

unchanged for different values of x, y, and r corresponding to different points on the terminal side OP of the angle θ.

From the properties of an isosceles right triangle, or an equilateral triangle, we see that a 30°, 45°, or 60° angle has the properties indicated in Fig. 24.

In this article we shall illustrate only the use of sine and cosine, since these are the functions primarily used in this chapter.

From Fig. 24 and the definitions of sine and cosine, it follows that:

$$\cos 30^\circ = \frac{\sqrt{3}}{2}; \qquad \cos 45^\circ = \frac{1}{\sqrt{2}}; \qquad \cos 60^\circ = \frac{1}{2};$$

$$\sin 30^\circ = \frac{1}{2}; \qquad \sin 45^\circ = \frac{1}{\sqrt{2}}; \qquad \sin 60^\circ = \frac{\sqrt{3}}{2}.$$

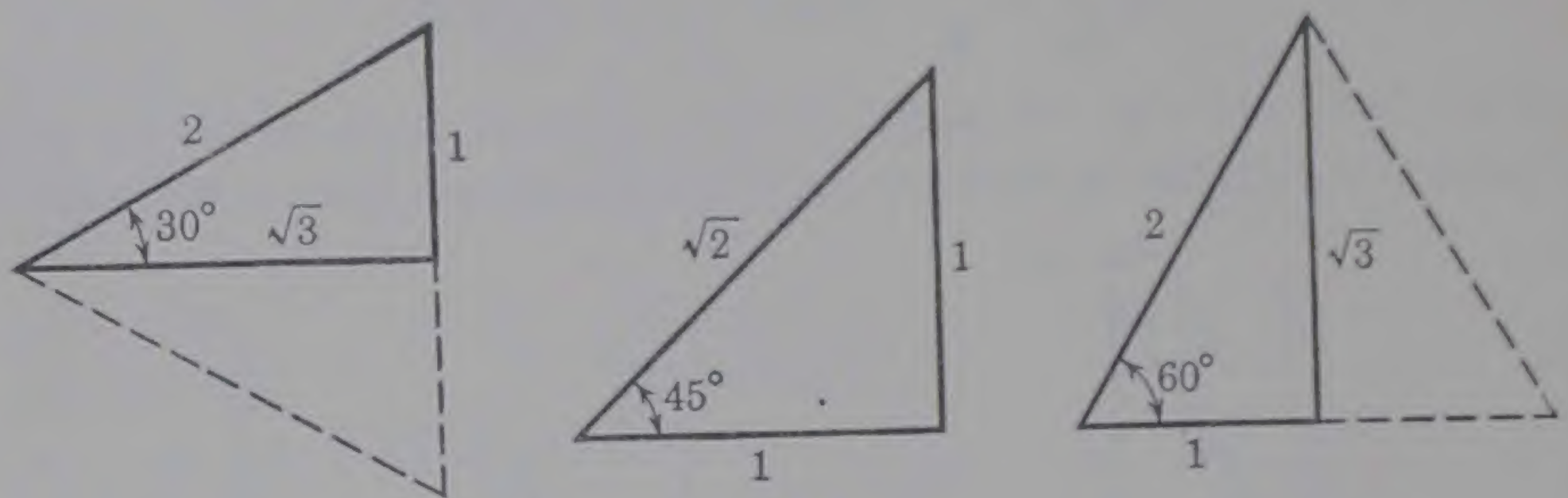


FIG. 24.

The following complex numbers can be written in polar form by using the foregoing information.

$$\frac{\sqrt{3}}{2} + i\frac{1}{2} = \cos 30^\circ + i \sin 30^\circ.$$

$$1 + i = \sqrt{2}(\cos 45^\circ + i \sin 45^\circ).$$

$$4 + 4i\sqrt{3} = 8(\cos 60^\circ + i \sin 60^\circ).$$

The properties of a 30°, 45°, or 60° angle can be extended to other angles. In the accompanying illustrations it should not be difficult to see the properties of the indicated angles, and their application to represent certain complex numbers in polar form.

ILLUSTRATION 1.

$$\cos 150^\circ = -\frac{\sqrt{3}}{2} \qquad \text{and} \qquad \sin 150^\circ = \frac{1}{2}.$$

Thus,

$$-\sqrt{3} + i = 2(\cos 150^\circ + i \sin 150^\circ).$$

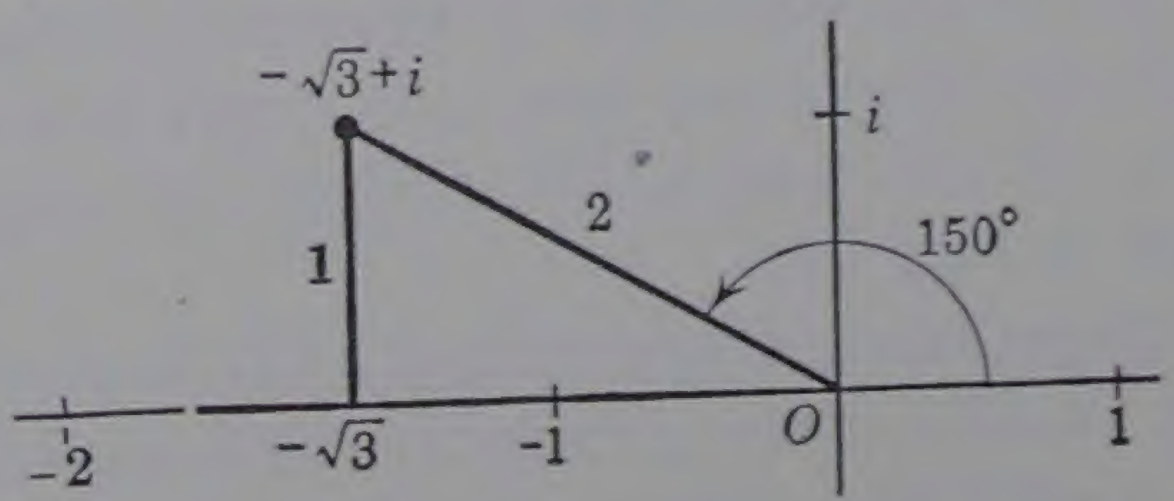


ILLUSTRATION 1.

ILLUSTRATION 2.

$$\cos 225° = -\frac{1}{\sqrt{2}} \qquad \text{and} \qquad \sin 225° = -\frac{1}{\sqrt{2}}.$$

Thus,

$$-2 - 2i = 2\sqrt{2}(\cos 225° + i \sin 225°).$$

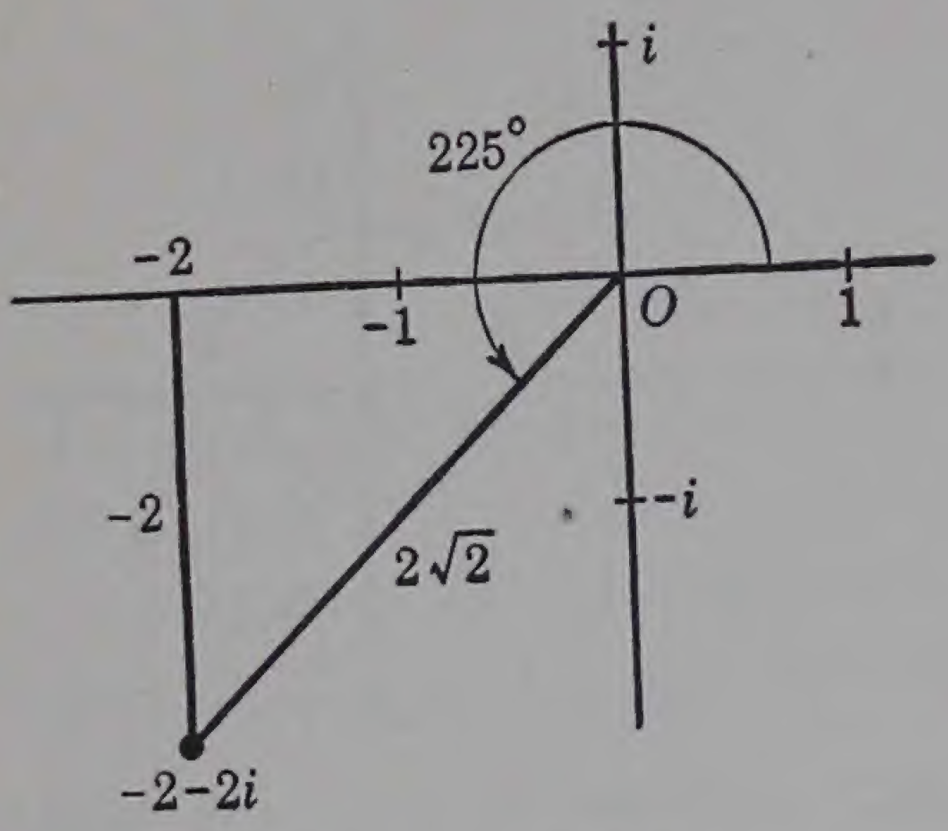


ILLUSTRATION 2.

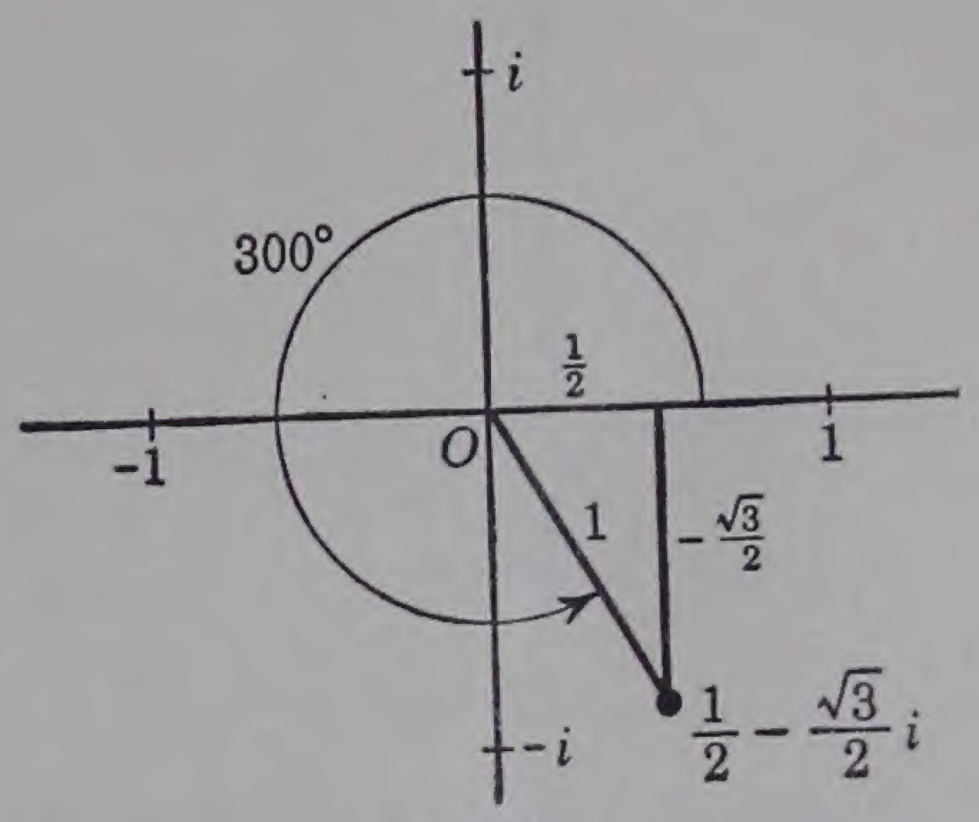


ILLUSTRATION 3.

ILLUSTRATION 3.

$$\cos 300° = \frac{1}{2} \qquad \text{and} \qquad \sin 300° = -\frac{\sqrt{3}}{2}.$$

Thus,

$$\frac{1}{2} - \frac{\sqrt{3}}{2} i = \cos 300° + i \sin 300°.$$

ILLUSTRATION 4. For 0°, we can consider a point P with coördinates (1, 0). Then $x = 1$, $y = 0$, and $r = 1$, and we have

$$\cos 0° = \frac{x}{r} = \frac{1}{1} = 1 \qquad \text{and} \qquad \sin 0° = \frac{y}{r} = \frac{0}{1} = 0.$$

Thus,

$$4 = 4(\cos 0° + i \sin 0°).$$

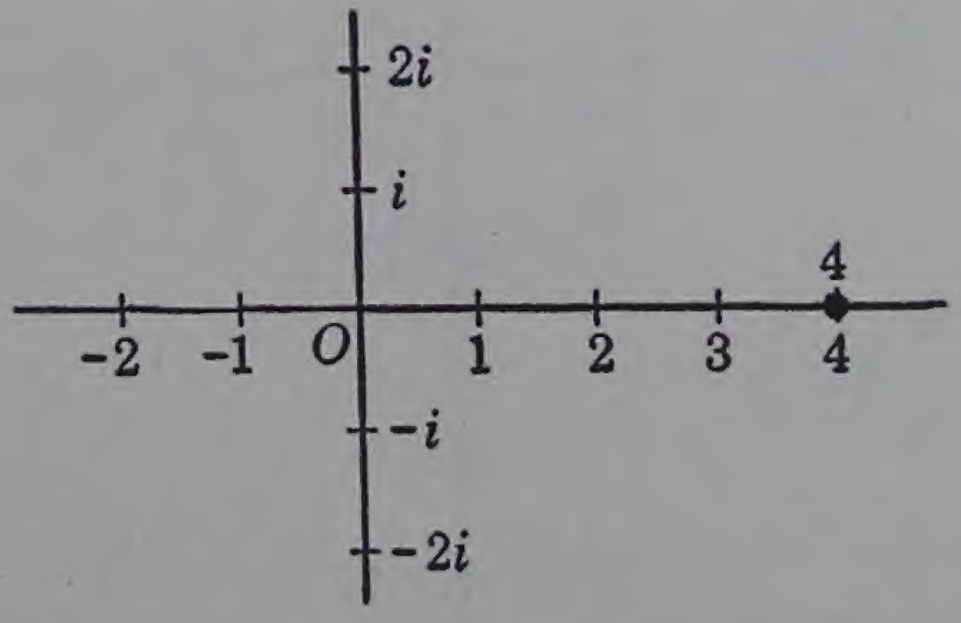


ILLUSTRATION 4.

ILLUSTRATION 5. For 90°, we can consider a point P with coördinates (0, 1). Then $x = 0$, $y = 1$, and $r = 1$, and we have

$$\cos 90° = 1 \qquad \text{and} \qquad \sin 90° = 0.$$

Thus,

$$3i = 3(\cos 90° + i \sin 90°).$$

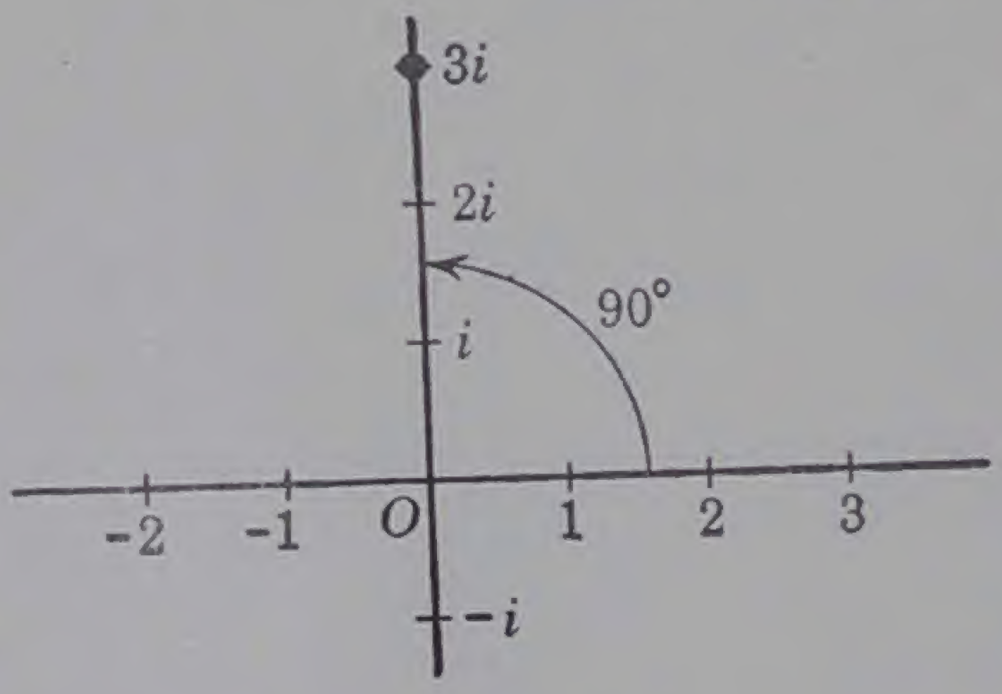


ILLUSTRATION 5.

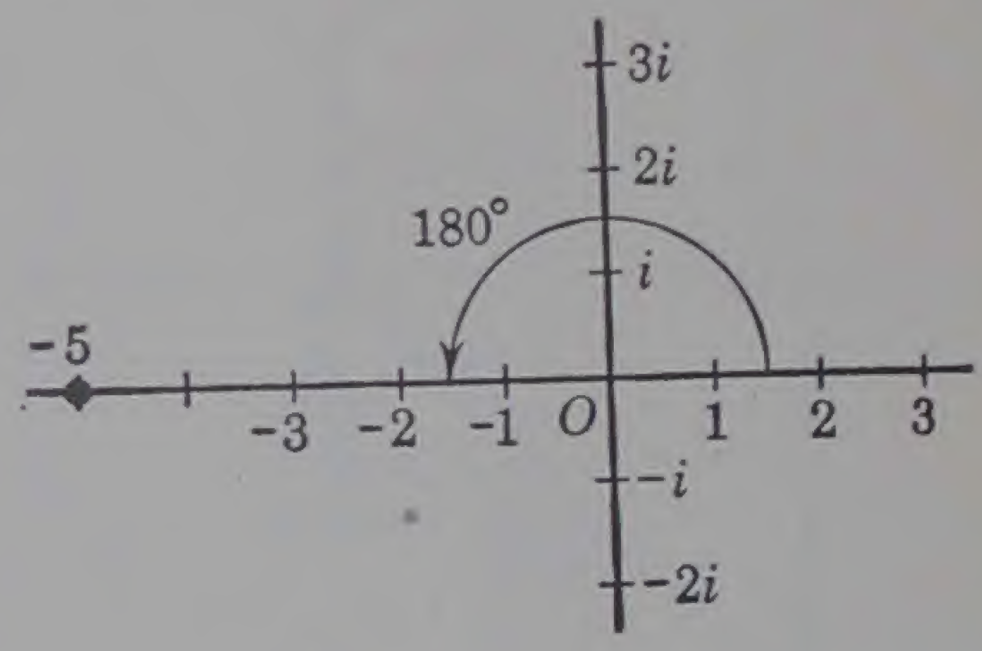


ILLUSTRATION 6.

ILLUSTRATION 6. For 180°, we can consider a point P with coördinates $(-1, 0)$. Then $x = -1$, $y = 0$, and $r = 1$, and we have

$$\cos 180° = -1 \qquad \text{and} \qquad \sin 180° = 0.$$

Thus,

$$-5 = 5(\cos 180° + i \sin 180°).$$

ILLUSTRATION 7. For 270°, we can consider a point P with coördinates $(0, -1)$. Then $x = 0$, $y = -1$, and $r = 1$, and we have

$$\cos 270° = 0 \qquad \text{and} \qquad \sin 270° = -1.$$

Thus,

$$-4i = 4(\cos 270° + i \sin 270°).$$

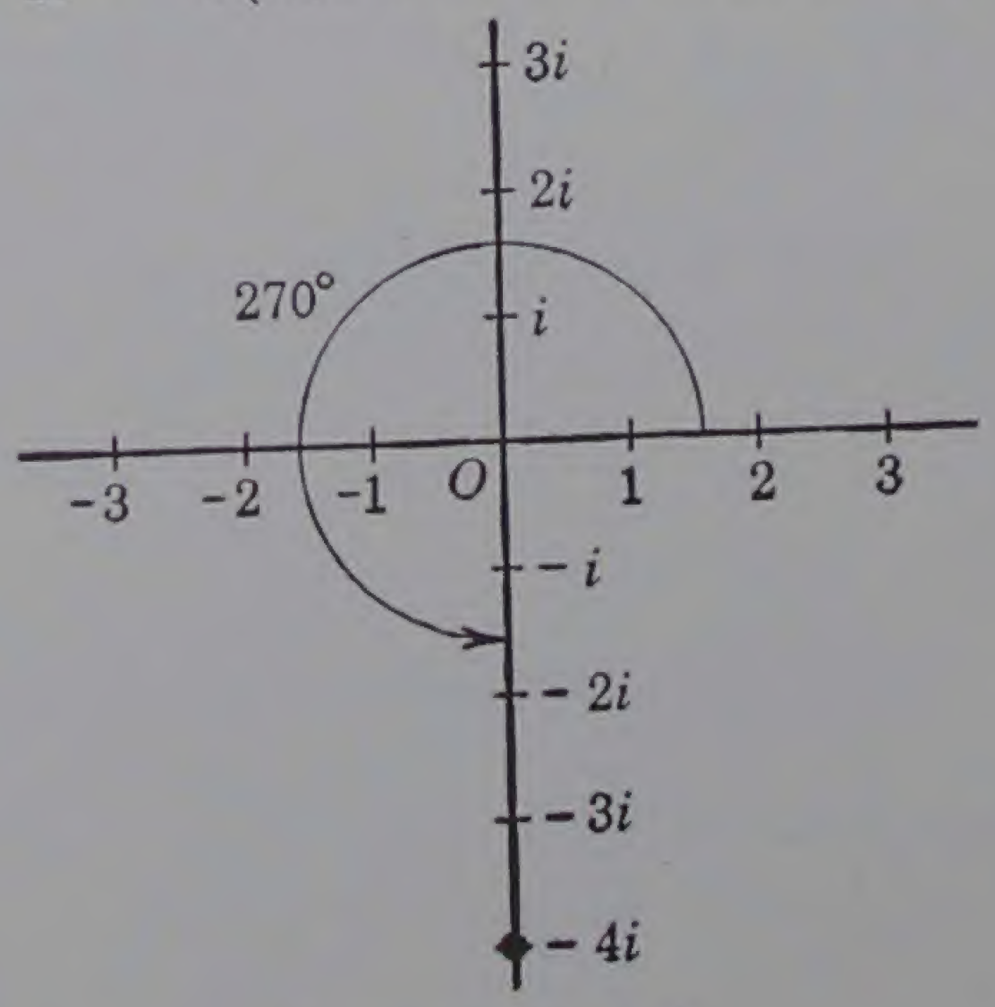


ILLUSTRATION 7.

The student should verify each of the following.

$$\sqrt{3} + i = 2(\cos 30° + i \sin 30°).$$
$$1 + i = \sqrt{2}\,(\cos 45° + i \sin 45°).$$
$$-1 + i = \sqrt{2}\,(\cos 135° + i \sin 135°).$$
$$1 - i\sqrt{3} = 2(\cos 300° + i \sin 300°).$$
$$\frac{\sqrt{3}}{2} + \frac{i}{2} = \cos 30° + i \sin 30°.$$

59. de MOIVRE'S THEOREM

It follows from the law for multiplying complex numbers (Equation 7) that the modulus of $[r(\cos\theta + i\sin\theta)]^n$ is $r \cdot r \cdot r \cdots$ r (n times), and the amplitude is $\theta + \theta + \theta + \cdots + \theta$ (n times). That is, if n is a positive integer

$$(a + bi)^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i \sin n\theta). \quad (13)$$

This relation is known as **de Moivre's theorem.**

The different powers of i can now be determined by de Moivre's theorem. To illustrate:

Since
$$i = \cos 90° + i \sin 90°,$$
then
$$i^n = \cos n \cdot 90° + i \sin n \cdot 90°.$$
Thus,
$$i^2 = \cos 180° + i \sin 180° = -1,$$
and
$$i^3 = \cos 270° + i \sin 270° = -i,$$
and
$$i^4 = \cos 360° + i \sin 360° = 1,$$

and so on (see Fig. 17).

It also follows from the law for multiplying complex numbers that

$$(a + bi)^{1/n} = [r(\cos\theta + i\sin\theta)]^{1/n}$$
$$= r^{1/n}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right). \quad (14)$$

When $a + bi$ is written in the polar form $r(\cos\theta + i\sin\theta)$, it is understood that θ is a positive angle less than 360°. However,

any angle that is formed by increasing θ by any number of complete revolutions can be used. Thus, it is possible to write

$$a + bi = r[\cos(\theta + k \cdot 360°) + i \sin(\theta + k \cdot 360°)], \quad (15)$$

in which k is any integer; then, by Equation (14),

$$(a + bi)^{1/n} = r^{1/n}\left[\cos\frac{\theta + k \cdot 360°}{n} + i \sin\frac{\theta + k \cdot 360°}{n}\right] \quad (r \neq 0)$$

$$= r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{360°}{n}k\right) + i \sin\left(\frac{\theta}{n} + \frac{360°}{n}k\right)\right], \quad (16)$$

in which $k = 0, 1, 2, 3, \cdots, n - 1$. As k takes on all integral values from 0 to $n - 1$, Equation (16) gives the n distinct nth roots of $a + bi$. If, for example, $k = n$, then

$$\frac{\theta + k \cdot 360}{n} = \frac{\theta}{n} + 360,$$

which gives the same root as when $k = 0$. Likewise, any other integral value of k greater than $n - 1$ will give one of the n roots obtained in Equation (16). Equation (16) is another form of de Moivre's theorem. To illustrate this theorem:

Example 1. Find the three cube roots of unity.

Solution. Writing 1 as a complex number in polar form, we have

$$1 = \cos 0° + i \sin 0°.$$

By de Moivre's theorem, Equation (16), the cube roots of 1 are

$$\cos\frac{k \cdot 360°}{3} + i \sin\frac{k \cdot 360°}{3}. \qquad (k = 0, 1, 2)$$

We shall represent the three cube roots of unity by 1, ω, and ω^2.

$$\text{For } (k = 0), \qquad 1 = \cos 0° + i \sin 0° = 1,$$

$$\text{For } (k = 1), \qquad \omega = \cos 120° + i \sin 120° = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

$$\text{For } (k = 2), \qquad \omega^2 = \cos 240° + i \sin 240° = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Example 2. Solve the equation $x^4 - 16 = 0$.

Solution. The roots of this equation are the four fourth roots of 16. Writing 16 as a complex number in polar form, we have

$$16 = 16(\cos 0° + i \sin 0°).$$

By de Moivre's theorem, Equation (16), the fourth roots of 16 are

$$16^{1/4}\left(\cos\frac{k\cdot 360^\circ}{4}+i\sin\frac{k\cdot 360^\circ}{4}\right). \qquad (k=0,\ 1,\ 2,\ 3)$$

For $(k=0)$, $\sqrt[4]{16}=2(\cos 0^\circ+i\sin 0^\circ)=2$,
For $(k=1)$, $\sqrt[4]{16}=2(\cos 90^\circ+i\sin 90^\circ)=2i$,
For $(k=2)$, $\sqrt[4]{16}=2(\cos 180^\circ+i\sin 180^\circ)=-2$,
For $(k=3)$, $\sqrt[4]{16}=2(\cos 270^\circ+i\sin 270^\circ)=-2i$.

Thus, the four fourth roots of 16 are $2,\ 2i,\ -2$, and $-2i$.

Since the polar form of any positive number a is $a(\cos 0^\circ+i\sin 0^\circ)$, it follows from Equation (16) that the root resulting from setting $k=0$ has the amplitude zero and hence must fall on the positive axis of reals. Thus, in Article 20, Chapter III, it was possible to define $a^{1/n}$ to be this positive root.

The polar form of any negative number $-a$ is $a(\cos 180^\circ+i\sin 180^\circ)$, hence $\theta=180^\circ$. Let us consider the nth root of $-a$ resulting from setting $k=\dfrac{n-1}{2}$ in Equation (16). Hence,

$$\frac{\theta+k\cdot 360^\circ}{n}=\frac{180^\circ+k\cdot 360^\circ}{n}=\frac{180^\circ(1+2k)}{n}=180^\circ,$$

since $1+2k=n$. Thus, we see that one of the n nth roots of a negative number $-a$ has the amplitude 180°, and hence falls on the negative axis of reals whenever n is odd $(n=2k-1)$. This is why, in Article 20, it was possible to define $(-a)^{1/n}$ to be $-a^{1/n}$ for n odd.

However, in case n is even, the amplitude $\dfrac{180^\circ(1+2k)}{n}$ for any of the nth roots of a negative number can have neither the value 0° nor the value 180° for any integral value of k. This is why, in Article 20, we did not define $(-a)^{1/n}$ to be any particular root for n even.

It follows from Equation (16) that all the n roots of any complex number have the same modulus, and hence can be represented graphically by points on the circumference of a circle drawn in the complex plane with center at the origin and radius equal to the principal nth root of the modulus of the given number. Furthermore, these n roots will be uniformily spaced on this circle,

the one root being at an angle of $\frac{\theta}{n}$ degrees from the axis of reals, and the remaining roots spaced at intervals of $\frac{360}{n}$ degrees from this root.

From the foregoing discussion it follows that the two square roots of 4, the three cube roots of 8, and the four fourth roots of 16 can be represented graphically by points on the circumference of a circle of radius 2 drawn in the complex plane as is shown in Fig. 25. The complex numbers corresponding to each of these points are the respective roots. The numbers are given for each of these points.

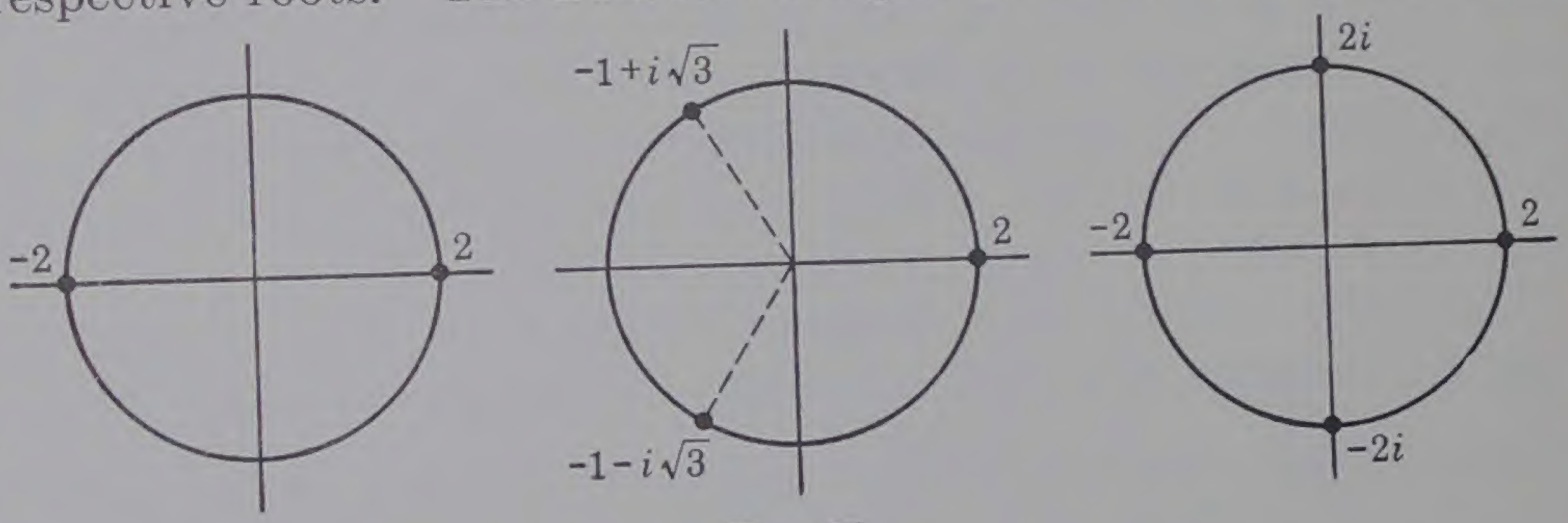


FIG. 25.

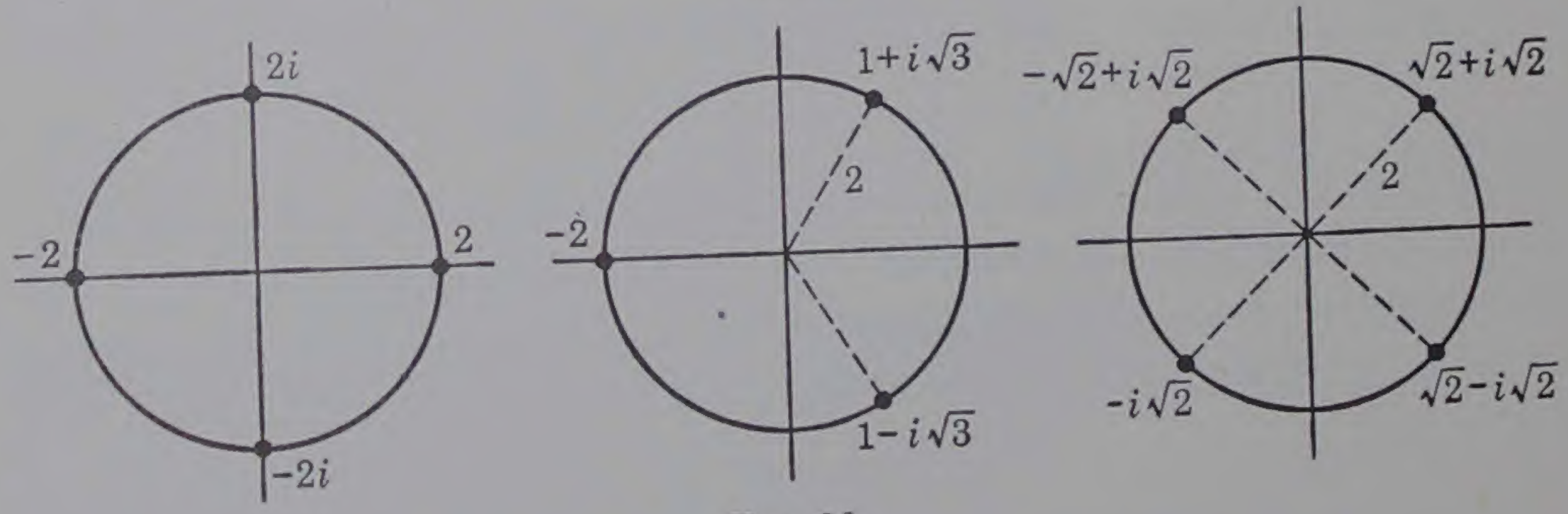


FIG. 26.

Observe that the three cube roots of 8 can be obtained by multiplying the three cube roots of unity (see Example 1) by 2, where 2 is the principal cube root of 8. The three cube roots of 8 are therefore,

$$2, \qquad 2\omega = -1 + i\sqrt{3}, \qquad \text{and} \qquad 2\omega^2 = -1 - i\sqrt{3}.$$

Likewise, the n nth roots of a can be obtained by multiplying the n nth roots of unity by $a^{1/n}$.

The two square roots of -4, the three cube roots of -8, and the four fourth roots of -16 can be represented graphically by points on the circumference of a circle of radius 2 as shown in Fig. 26.

Example 3. Find the six sixth roots of $1 - i\sqrt{3}$.

Solution. The polar form of $1 - i\sqrt{3}$ is $2(\cos 300° + i \sin 300°)$. By de Moivre's theorem:

$$(1 - i\sqrt{3})^{1/6} = \{2[\cos (300° + k360°) + i \sin (300° + k360°)]\}^{1/6}$$
$$= 2^{1/6}[\cos (50° + k \cdot 60°) + i \sin (50° + k \cdot 60°)].$$

For $(k = 0)$, $(1 - i\sqrt{3})^{1/6} = 2^{1/6}(\cos 50° + i \sin 50°)$,
For $(k = 1)$, $(1 - i\sqrt{3})^{1/6} = 2^{1/6}(\cos 110° + i \sin 110°)$,
For $(k = 2)$, $(1 - i\sqrt{3})^{1/6} = 2^{1/6}(\cos 170° + i \sin 170°)$,
For $(k = 3)$, $(1 - i\sqrt{3})^{1/6} = 2^{1/6}(\cos 230° + i \sin 230°)$,
For $(k = 4)$, $(1 - i\sqrt{3})^{1/6} = 2^{1/6}(\cos 290° + i \sin 290°)$,
For $(k = 5)$, $(1 - i\sqrt{3})^{1/6} = 2^{1/6}(\cos 350° + i \sin 350°)$.

The value of each of these sixth roots of $1 - i\sqrt{3}$ can be found with the aid of a table of natural trigonometric functions. Likewise, the polar form of any complex number $a + bi$ can be found with the aid of a table of natural trigonometric functions, since $\tan \theta = \frac{b}{a}$.

Heron of Alexandria, in the first century A.D., gives the oldest expression for imaginary roots. When solving a quadratic equation, he obtained the expression $\sqrt{81 - 144}$, but in its stead he evaluated $\sqrt{144 - 81}$. The Italian H. Cardan (1501–1576), in 1545, was the first mathematician to retain the square root of a negative number in the solution of a quadratic equation. He gave the solution of $x(10 - x) = 40$ as $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$, and was able to prove by multiplication that his results were correct. In 1631, Oughtred called negative numbers imaginary numbers, but Descartes, in 1637, was the first to call the square root of a negative number an imaginary number. In 1777, the Swiss mathematician Leonhard Euler (1707–1783), introduced i for $\sqrt{-1}$, and in 1821, the French mathematician A. L. Cauchy (1789–1857) introduced the term conjugate for $a + bi$ and $a - bi$.

EXERCISES

Simplify the expressions in each of the following twelve exercises. Express the results in rectangular form.

1. $2(\cos 15° + i \sin 15°) \cdot 4(\cos 30° + i \sin 30°)$.
2. $4(\cos 72° + i \sin 72°) \cdot 5(\cos 48° + i \sin 48°)$.
3. $6(\cos 170° + i \sin 170°) \cdot 2(\cos 130° + i \sin 130°)$.
4. $3(\cos 75° + i \sin 75°) \cdot 4(\cos 105° + i \sin 105°)$.

5. $8(\cos 80° + i \sin 80°) \div 2(\cos 20° + i \sin 20°)$.
6. $12(\cos 190° + i \sin 190°) \div 3(\cos 70° + i \sin 70°)$.
7. $6(\cos 35° + i \sin 35°) \div 2(\cos 95° + i \sin 95°)$.
8. $10(\cos 120° + i \sin 120°) \div 5(\cos 330° + i \sin 330°)$.
9. $[2(\cos 70° + i \sin 70°)]^3$.
10. $[3(\cos 80° + i \sin 80°)]^3$.
11. $[\frac{1}{2}(\cos 50° + i \sin 50°)]^6$.
12. $[2(\cos 33° + i \sin 33°)]^{10}$.

Express each of the following numbers (Exercises 13 through 24) in polar form and then perform the indicated operations. Express the results in rectangular form.

13. $(1 + i)^3$.
14. $(1 - i\sqrt{3})^4$.
15. $\left(\frac{-\sqrt{3}}{2} + \frac{i}{2}\right)^5$.
16. $(3 - 3i)^4$.
17. $(\sqrt{3} - i)(-1 - i\sqrt{3})$.
18. $(-\sqrt{3} - i)(1 + i\sqrt{3})$.
19. $\frac{i}{1 + i}$.
20. $\frac{-2}{-\sqrt{3} + i}$.
21. $(-1 + i\sqrt{3})^{-3}$.
22. $\frac{(-1 + i\sqrt{3})}{-\sqrt{3} + i}$.
23. $\frac{(8\sqrt{3} - 8i)}{1 - i\sqrt{3}}$.
24. $\frac{(-1 - i\sqrt{3})}{2 - 2i\sqrt{3}}$.

By de Moivre's theorem, find the indicated roots of each of the following (Exercises 25 through 32), leaving answers in polar form when the results cannot be simplified without use of tables.

25. The two square roots of i.
26. The two square roots of $-i$.
27. The three cube roots of 8.
28. The three cube roots of $8i$.
29. The four fourth roots of $-8 + 8i\sqrt{3}$.
30. The four fourth roots of $-8 - 8i\sqrt{3}$.
31. The five fifth roots of $32i$.
32. The six sixth roots of $-\sqrt{3} + i$.

Show graphically all the values of x which will satisfy each of the following equations. Draw to scale and give each value of x. Then find all the roots by de Moivre's theorem.

33. $x^3 = 1$.
34. $x^4 = 1$.
35. $x^4 = -1$.
36. $x^6 = 1$.
37. $x^3 = 27$.
38. $x^8 = 256$.
39. $x^4 = -81$.
40. $x^2 = i$.
41. $x^3 = -27$.
42. $x^4 = 81$.
43. $x^2 = -i$.
44. $x^3 = \frac{1 + i}{\sqrt{2}}$.

CHAPTER XI

Quadratic Equations

METHODS FOR SOLVING QUADRATIC EQUATIONS

1. *Factoring.*
2. *Completing the Square.*
3. *Formula.*
4. *Graphing.*

60. QUADRATIC EQUATIONS

The standard form of a quadratic equation is

$$f(x) = a_0x^2 + a_1x + a_2 = 0,$$

but for convenience it is usually written as

$$ax^2 + bx + c = 0, \tag{1}$$

where a is any constant except zero, and b and c are any constants whatsoever.

There are four methods for solving quadratic equations: *by factoring, by completing the square, by formula, and by graphing.*

Factoring

The method of factoring is used to solve quadratic equations whose roots are rational.

The method of solving equations by factoring is based upon the following fundamental principle:

If the product of several numbers is zero, then at least one of the numbers must be zero.

Example 1. Solve the equation $2x^2 + 5x - 12 = 0$.

Solution. Given

$$2x^2 + 5x - 12 = 0.$$

Factoring, we have

$$(2x - 3)(x + 4) = 0.$$

The only values x can have are those values which will make one of the two numbers $2x - 3$ or $x + 4$ equal to zero. The value of x which will

make $2x - 3$ equal to zero is found by solving the linear equation

$$2x - 3 = 0.$$

This value is

$$x = \tfrac{3}{2}.$$

The value of x which will make $x + 4$ equal to zero is found by solving the linear equation

$$x + 4 = 0.$$

This value is

$$x = -4.$$

Hence, the roots or solutions of the quadratic equation $2x^2 + 5x - 12 = 0$ are

$$x = \tfrac{3}{2} \quad \text{and} \quad x = -4.$$

Check: If $x = \frac{3}{2}$, then $2 \cdot \frac{9}{4} + 5 \cdot \frac{3}{2} - 12 = \frac{9}{2} + \frac{15}{2} - 12 = 0$. If $x = -4$, then $2 \cdot 16 + 5(-4) - 12 = 32 - 20 - 12 = 0$.

The equation $(2x - 3)(x + 4) = 0$ merely states that the product of the two numbers $2x - 3$ and $x + 4$ is equal to zero. This can be true only when one, or both, of these numbers are zero. *Hence, a zero of one factor is a solution of the equation.*

It should be observed that *if the product of two or more factors is equal to some number different from zero, then nothing definite is known about each factor separately.* To illustrate, suppose we write $2x^2 + 5x - 12 = 0$ as

$$2x^2 + 5x + 3 = 15;$$

then we obtain, by factoring,

$$(2x + 3)(x + 1) = 15.$$

This equation merely states that the product of the two numbers $2x + 3$ and $x + 1$ is equal to 15. Since there is an unlimited number of pairs of numbers whose product is 15, we cannot infer from this fact alone the value of the separate factors.

Hence, a quadratic equation cannot be solved by factoring, unless the right-hand member of the equation is zero. The roots or solutions of the equation are the zeros of the two linear factors; that is, they are the roots of the two linear equations formed by setting each of the linear factors equal to zero.

If $b = 0$, that is, if there is no term containing the first power of the unknown, the roots of such equations are conveniently obtained by first solving for the square of the unknown. Then by extracting the square root of both members of the equation we have the two solutions immediately. For example, if $b = 0$, the standard form of the equation becomes

$$ax^2 + c = 0.$$

Solving for x^2, gives

$$x^2 = -\frac{c}{a};$$

extracting the square root gives

$$x = \pm\sqrt{-\frac{c}{a}}.$$

There are always two square roots of any number. The one square root is positive and the other square root is negative.

Example 2. Solve $x^2 - 25 = 0$.

Solution. Transposing, we have $x^2 = 25$;

then, $x = +5$ or $x = -5$,

but we usually write it as $x = \pm 5$.

That is, both $(5)^2$ and $(-5)^2$ are equal to 25.

Graphing

If the roots of the quadratic equation $ax^2 + bx + c = 0$ are real, then they can be obtained by finding the x-intercepts of the graph of $y = ax^2 + bx + c$. That is, *the real roots of* $ax^2 + bx + c = 0$ *are the abscissas of the points where the graph of* $y = ax^2 + bx + c$ *crosses or touches the x-axis.* We shall assume a, b, and c to be real numbers.

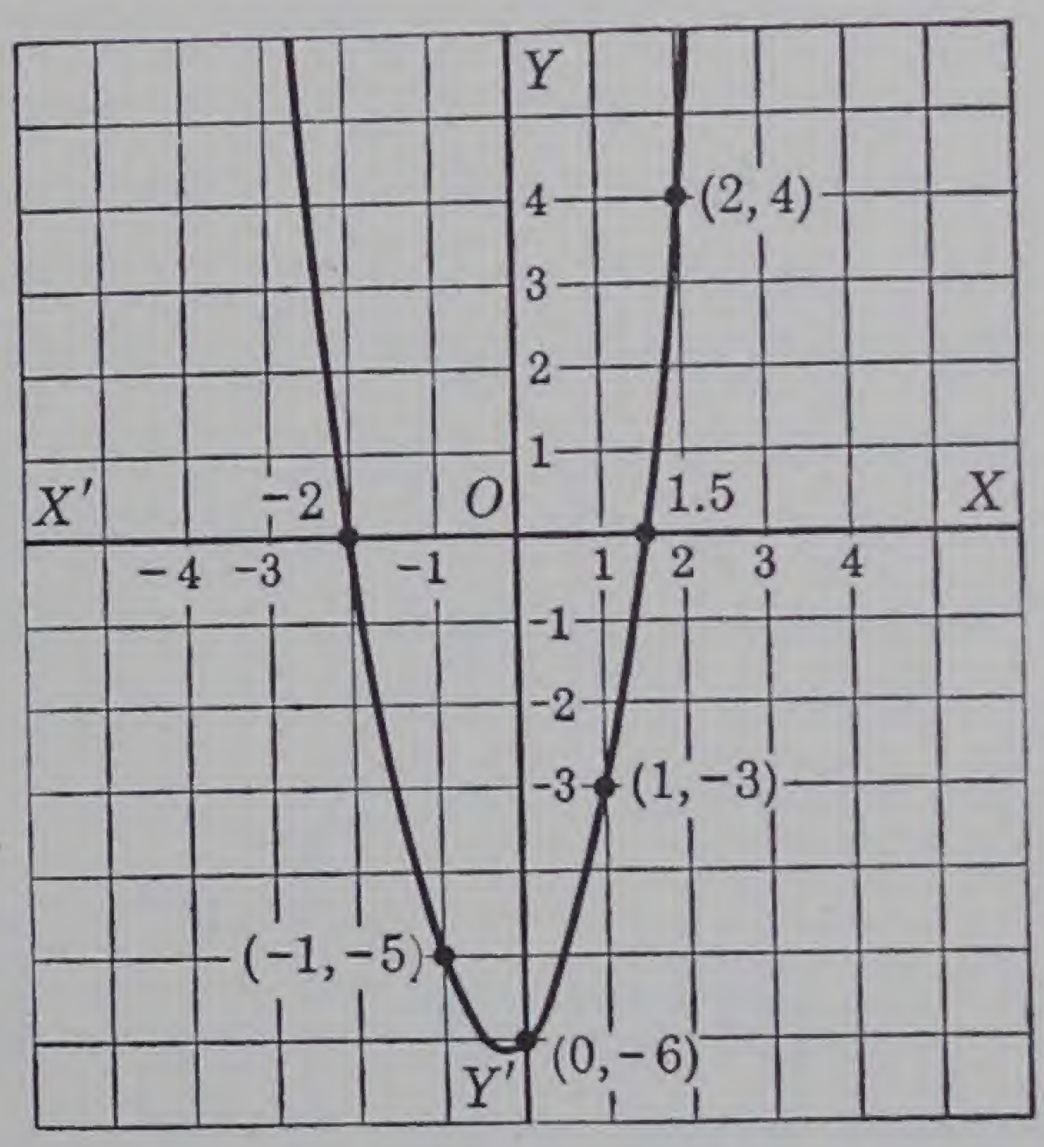


FIG. 27.

Example 3. Solve $2x^2 + x - 6 = 0$ by graph.

Solution. Compute the accompanying table of points from the equation $y = 2x^2 + x - 6$. Plot these points and draw a smooth curve through the plotted points as in Fig. 27.

x	y
−3	9
−2	0
−1	−5
0	−6
1	−3
2	4
3	15

The graph crosses the x-axis at $x = -2$ and $x = 1.5$, which are the roots of the given equation $2x^2 + x - 6 = 0$.

If the graph of $y = ax^2 + bx + c$ crosses the x-axis, then the roots of the equation $ax^2 + bx + c = 0$ are real and unequal. If the graph just touches the x-axis, then both roots are equal. If the graph neither crosses nor touches the x-axis, then both roots are imaginary

EXERCISES

Solve each of the following eighteen equations by factoring, and show the solutions by graphs.

1. $x^2 - x - 2 = 0$.
2. $2x^2 + 5x - 3 = 0$.
3. $x^2 - 10x + 25 = 0$.
4. $x^2 - 5x + 6 = 0$.
5. $8x^2 + 18x + 9 = 0$.
6. $10 - 3x - x^2 = 0$.
7. $x^2 + 8x + 15 = 0$.
8. $x^2 - ax - 6a^2 = 0$.
9. $x^2 - 16 = 0$.
10. $3x^2 - 5 = 0$.
11. $6x^2 - 7x = 0$.
12. $(3x - 2)^2 = 9$.
13. $5 = 3x^2 + 2x$.
14. $(2x + 1)^2 = 9$.
15. $9y^2 - 5y = 0$.
16. $6y^2 - y - 12 = 0$.
17. $2x^2 - 5x - 7 = 0$.
18. $2x^2 + ax - 6a^2 = 0$.

Completing the Square

Every quadratic equation can be solved by the method of completing the square.

If a quadratic equation does not factor readily into its linear factors, we may employ a method called **completing the square** to solve the equation. First, we determine a constant to add to both members of the equation so as to form a perfect square with the two terms containing the square of the unknown and the first power of the unknown. Then, by taking the square root of both members of the resulting equation, we have two linear equations, from which the two roots of the given quadratic equation can be obtained. That such a constant exists is at once apparent from the following identity:

$$(x + a)^2 = x^2 + 2ax + a^2.$$

The constant a^2 which must be added to $x^2 + 2ax$ to form a perfect square is the square of $\frac{1}{2}$ of $2a$, where $2a$ is the coefficient of x. Hence,

If the coefficient of x^2 is 1, then the constant necessary to complete the square is the square of one-half the coefficient of x. To illustrate:

Example 4. Solve the equation $2x^2 + 5x - 12 = 0$ by completing the square.

Solution. Transposing, we have

$$2x^2 + 5x = 12;$$

dividing by 2 gives

$$x^2 + \tfrac{5}{2}x = 6.$$

Add to both members of the equation $(\frac{5}{4})^2$, the square of one-half the coefficient of x. It is best to add to the left-hand member of the equation the indicated square, but add to the right-hand member of the equation the number squared. Thus,

$$x^2 + \tfrac{5}{2}x + (\tfrac{5}{4})^2 = 6 + \tfrac{25}{16},$$

or

$$(x + \tfrac{5}{4})^2 = \tfrac{121}{16}.$$

Extracting the square root of both members of the equation, we have the two linear equations

$$x + \tfrac{5}{4} = \pm \tfrac{11}{4};$$

that is,

$$x + \tfrac{5}{4} = \tfrac{11}{4} \qquad \text{and} \qquad x + \tfrac{5}{4} = -\tfrac{11}{4}.$$

Solving each equation for x, we have the two roots

$$x = \tfrac{3}{2} \qquad \text{and} \qquad x = -4.$$

Check: If $x = \frac{3}{2}$, then $2(\frac{3}{2})^2 + 5(\frac{3}{2}) - 12 = \frac{9}{2} + \frac{15}{2} - 12 = 0$. If $x = -4$, then $2(-4)^2 + 5(-4) - 12 = 32 - 20 - 12 = 0$.

Formula

Every quadratic equation can be solved by formula.

Let us apply the method of completing the square to the general quadratic equation

$$ax^2 + bx + c = 0; \tag{2}$$

transposing c and dividing by a gives

$$x^2 + \frac{b}{a}x = -\frac{c}{a}. \tag{3}$$

The constant necessary to make the left-hand member of the equation a perfect square is $\left(\frac{b}{2a}\right)^2$. Adding this to both members of the equation, we have

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}; \tag{4}$$

then

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}. \tag{5}$$

Extracting the square root of both members of the equation, we have the two linear equations

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \tag{6}$$

or

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x + \frac{b}{2a} = \frac{-\sqrt{b^2 - 4ac}}{2a}. \tag{7}$$

Hence, the two roots of the general quadratic equation

$$ax^2 + bx + c = 0$$

are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{8}$$

These two roots are usually written as:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{9}$$

The expression $\mathbf{b^2 - 4ac}$ is called the **discriminant** of the equation $ax^2 + bx + c = 0$.

Since Equation (9) gives the two roots of the general quadratic equation $ax^2 + bx + c = 0$, *we may use it as a* ***formula*** *for the solution of all quadratic equations, where proper values are assigned to* $\boldsymbol{a}$, $\boldsymbol{b}$, *and* $\boldsymbol{c}$.

It is evident from the formula given in Equation (9) that the roots of a quadratic equation are rational if $b^2 - 4ac$ is a perfect square. Hence, *a quadratic equation can always be solved by factoring if* $\boldsymbol{b^2 - 4ac}$ *is a perfect square.*

All quadratic equations in which $\boldsymbol{b^2 - 4ac}$ *is not a perfect square must be solved either by completing the square or by formula.*

When the quadratic equation is written in standard form, $\boldsymbol{a}$ *is the coefficient of the square of the unknown,* $\boldsymbol{b}$ *is the coefficient of the unknown, and* $\boldsymbol{c}$ *is the term free of the unknown.* To illustrate, find a, b, and c for the following equation:

$$3x^2 - kx + 5 = kx^2 - 3x + k.$$

Transposing, we have

$$3x^2 - kx^2 - kx + 3x + 5 - k = 0;$$

collecting gives

$$(3 - k)x^2 + (-k + 3)x + (5 - k) = 0,$$

whence

$$a = 3 - k; \qquad b = -k + 3; \qquad \text{and} \qquad c = 5 - k.$$

Observe that the standard form of a quadratic equation consists of only three terms.

Example 5. Solve the equation $2x^2 + x - 6 = 0$ by formula.

Solution. In this case

$$a = 2, \qquad b = 1, \qquad \text{and} \qquad c = -6.$$

Given the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

we obtain, by substituting,

$$x = \frac{-1 \pm \sqrt{1^2 - 4(2)(-6)}}{2 \cdot 2}.$$

Simplifying gives

$$x = \frac{-1 \pm \sqrt{49}}{4};$$

then

$$x = \frac{-1 + 7}{4} \qquad \text{and} \qquad x = \frac{-1 - 7}{4}.$$

Hence the roots are

$$x = \tfrac{3}{2} \qquad \text{and} \qquad x = -2.$$

Check: If $x = \frac{3}{2}$, then $2(\frac{3}{2})^2 + \frac{3}{2} - 6 = \frac{9}{2} + \frac{3}{2} - 6 = 0$. If $x = -2$, then $2(-2)^2 + (-2) - 6 = 8 - 2 - 6 = 0$.

A second method for completing the square, if the coefficient of x^2 is different from 1, is apparent from the following. If we multiply each term of the quadratic equation

$$ax^2 + bx = -c \tag{10}$$

by $4a$, we have

$$4a^2x^2 + 4abx = -4ac. \tag{11}$$

Then it is evident that the constant b^2 is all that is necessary for completing the square. Thus, after adding b^2 to both members of the equation, we have

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac, \tag{12}$$

or

$$(2ax + b)^2 = b^2 - 4ac. \tag{13}$$

Extracting the square root of both members of Equation (13) gives

$$2ax + b = \pm \sqrt{b^2 - 4ac}. \tag{14}$$

Solving for x gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{15}$$

We can complete the square by multiplying each term of the given equation by $4a$ *(that is, by* 4 *times the coefficient of* x^2*), and then adding* b^2 *(that is, the square of the coefficient of* x *in the original equation) to both members of the equation.* To illustrate:

Example 6. Solve the equation $2x^2 + x - 6 = 0$.

Solution. Multiplying by 8 and transposing, we get

$$16x^2 + 8x = 48;$$

adding 1^2 to both members gives

$$16x^2 + 8x + 1 = 48 + 1;$$

then

$$(4x + 1)^2 = 49;$$

extracting the square root gives

$$4x + 1 = \pm 7.$$

Hence, the roots are

$$x = \tfrac{3}{2} \quad \text{and} \quad x = -2.$$

Let us find the turning point of the graph of the general quadratic function

$$y = ax^2 + bx + c.$$

Differentiating gives

$$\frac{dy}{dx} = 2ax + b. \tag{16}$$

If we set the derivative equal to zero and solve for x, we have

$$2ax + b = 0 \tag{17}$$

or

$$x = -\frac{b}{2a}, \tag{18}$$

which is the abscissa of the turning point. The ordinate of the turning point is found by solving for y in $y = ax^2 + bx + c$, where $x = -\frac{b}{2a}$. Therefore

$$y = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c \tag{19}$$

$$= -\frac{b^2 - 4ac}{4a}. \tag{20}$$

Hence, the coördinates of the vertex of the parabola $y = ax^2 + bx + c$ are

$$\left(-\frac{b}{2a}, \quad -\frac{b^2 - 4ac}{4a}\right).$$

To illustrate, the coördinates of the vertex of the parabola in Fig. 27 for the equation $y = 2x^2 + x - 6$ are $(-\frac{1}{4}, -\frac{49}{8})$.

QUESTIONS

1. Define quadratic equation.
2. Give the standard form for a quadratic equation.
3. Give the meaning of a, of b, and of c in the standard quadratic equation.
4. Give four methods for solving a quadratic equation.

EXERCISES

Solve the following forty-eight equations by any method.

1. $x^2 + 6x + 7 = 0$.
2. $x^2 + 3x - 5 = 0$.
3. $3x^2 + 5x + 1 = 0$.
4. $ax^2 + bx = 0$.
5. $x^2 - 6x + 9 = -1$.
6. $x^2 = 12x + 35$.
7. $x^2 - 3x + 1 = 0$.
8. $(2x + 1)^2 = 9x$.
9. $2x^2 - x - 3 = 2$.
10. $5y^2 - 53y + 30 = 0$.
11. $3x^2 - 4x + 2 = 0$.
12. $(2x - 3)(x + 2) = 2x$.
13. $x^2 - 2x + 1 = 64$.
14. $18x^2 - 63x + 40 = 0$.
15. $ax(x - a) = x - a$.
16. $a^2x^2 + 2ax - 8 = 0$.
17. $ax^2 + c = 0$.
18. $x^2 + bx = ax + ab$.
19. $x^2 + ax = a + x$.
20. $cx^2 + bx + a = 0$.
21. $x^2 - ax + 2x - 2a = 0$.
22. $x^2 - a^2 = x - a$.
23. $2x^2 - 3xy - 20y^2 = 0$.
24. $x^2 - 5xy + y^2 = 0$.
25. $3x^2 - 2xy - 5y^2 = 0$.
26. $80x^2 - 122xy + 45y^2 = 0$.
27. $3x^2 - 5xy + 8y^2 = 0$.
28. $a^2bx^2 - a^3x = ab - b^2x$.
29. $(2x - 1)(x + 2) = 12$.
30. $6(x^2 - a) = x(9a - 4)$.
31. $x^2 + 2bx = a^2 - b^2$.
32. $x^2 + 4abx = (a^2 - b^2)^2$.
33. $x^2 + \sqrt{3}x - 1 = \sqrt{3}$.
34. $\frac{x}{a} - \frac{a}{x} = \frac{x}{a - b} - \frac{a - b}{x}$.
35. $x^2 - \frac{3}{4}x + \frac{2}{3} = 0$.
36. $\frac{x + 1}{x} - 1 = \frac{2x}{x + 1}$.
37. $\frac{x + 5}{2x + 1} + \frac{6 - x}{2 - x} = 1$.
38. $\frac{x}{3 - x} - \frac{3 - x}{x} = \frac{3}{2}$.
39. $\frac{1}{x} + x = \frac{1}{a} + a$.
40. $\frac{x}{a} + \frac{a}{x} = \frac{x}{b} + \frac{b}{x}$.
41. $\frac{x}{x - 1} - \frac{x - 1}{x} = 1$.
42. $\frac{a^2}{(x - b)^2} = \frac{b^2}{(x - a)^2}$.

43. $\dfrac{x+1}{x^2-4}+\dfrac{2}{5(2-x)}=\dfrac{x-1}{x+2}$.

44. $\dfrac{1}{4x+2}-\dfrac{1}{2x-1}-\dfrac{1}{1-4x^2}=\dfrac{1}{4}$.

45. $\dfrac{x^2+5x+11}{x^2+x-6}+\dfrac{1}{x+3}=4$.

46. $\dfrac{x+1}{x(x-1)}-\dfrac{1}{3x-3}+\dfrac{1}{2x}=-\dfrac{1}{3}$.

47. $\dfrac{3}{1-x}+\dfrac{6}{x-2}=\dfrac{2}{3-x}-\dfrac{5}{4-x}$.

48. $\dfrac{x+2}{x^2-1}-\dfrac{x-2}{x^2-x}+\dfrac{x+2}{x^2+x}=0$.

The following formulas (Exercises 49 through 60) can be found in any handbook for engineers. For each formula, solve for each of the letters in the formula in terms of the remaining letters in the formula.

49. $V=\pi r^2h$.

50. $\dfrac{3W(A-a)}{64{,}000B}=T$.

51. $g=\dfrac{R}{R^2+X^2}$.

52. $W=\dfrac{kbd^2}{L}$.

53. $S=\dfrac{n}{2}[2a+(n-1)d]$.

54. $A=2\pi rh+2\pi r^2$.

55. $m=\dfrac{3WDL^2}{4}$.

56. $t=2\pi\sqrt{\dfrac{l}{g}}$.

57. $s=vt+\dfrac{1}{2}gt^2$.

58. $z=\dfrac{t(3kd-2t)}{3(2kd-t)}$.

59. $s=\dfrac{mgl}{k\pi r^2}$.

60. $p=\dfrac{200(t+1)^2}{s-6}$.

61. In an arithmetic progression the sum of n terms is given by the formula $S=\dfrac{n}{2}[2a+(n-1)d]$. Find n if $a=5$, $d=2$, and $S=140$.

62. The product of what two consecutive integers is 182?

63. Find two consecutive integers whose product is 110.

64. What two numbers differing by 10 have a product of 875?

65. Find two consecutive integers the sum of whose squares is 113.

66. Twelve times the reciprocal of a number added to 5 equals twice the number. Find the number.

67. One side of a rectangle is 5 feet longer than the other side and the area is 150 square feet. Find its dimensions.

68. A rectangle whose length is 15 feet longer and whose breadth is 10 feet wider than the sides of a square, has an area three times the area of the square. Find the side of the square.

69. An automobile is traveling at an unknown rate. If it traveled 5 miles per hour faster, it would take 1 hour less to go 360 miles. Find the rate of the automobile. Find the rate if 5 miles per hour faster reduces the time by 2 hours and 24 minutes.

70. One automobile travels 10 miles per hour faster than another automobile and the faster one travels a distance of 690 miles in 4 hours and 10 minutes less time than it takes the slower automobile. Find the speed of each car.

71. A merchant sold a certain amount of goods for $600. His profit in per cent was equal to what the goods cost him in dollars. How much did the goods cost him?

72. A merchant sold a suit of clothes for $25, and his loss in per cent was equal to what the suit cost him in dollars. How much did the suit cost the dealer?

73. A man bought a number of eggs for $7.50, and sold all but 5 dozen for $7.50. If his profit is 5 cents per dozen, how many eggs did he buy?

74. A man contracted to do a piece of work for $85. It took him 3 days longer than he expected; he thus received 75 cents per day less than he expected. In how many days did he expect to complete the work?

75. A man wishes to make a sidewalk on the two sides of his corner lot, which is 50 feet by 140 feet. How wide can he make the walk if he wishes to make it 4 inches thick and has 325 cubic feet of gravel?

76. How wide a strip must be mowed around a rectangular field 300 yards long and 200 yards wide so as to mow $\frac{2}{3}$ of the field?

77. In a group of towns, each one is connected with every other town by a road, and no three towns are on the same road. If there are 55 roads, how many towns are there?

78. A and B working together can do a piece of work in 12 days. If it takes B 10 days longer to do the work than it takes A, how long does it take A to do the work?

79. A square piece of tin 14 inches wide is to have strips along the edges turned up vertically to form a gutter with a rectangular cross section. How deep should the gutter be to have the greatest carrying capacity? (See Article 41.)

80. A man walked a distance of 10 miles, and wished to return in 20 minutes less time than it took him to go. If this was accomplished by walking back 1 mile per hour faster than he went, find his rate of walking in each direction.

81. If a particle is thrown upward with a velocity of V_0 feet per second, the distance S from the ground to the particle after a given time t is

given by the formula $S = V_0t - \frac{1}{2}gt^2$, in which g may be taken as 32. If the particle is thrown downward with a velocity of V_0, the distance from the starting point to the particle after a given time t is given by the formula $S = V_0t + \frac{1}{2}gt^2$. In each case find the formula for the velocity V at any given time $\left(\frac{dS}{dt} = V\right)$.

82. If a particle is thrown upward with a velocity of 100 feet per second (see Exercise 81), (a) how high will it rise? (b) when will it strike the ground? (c) with what velocity will it strike the ground? (d) how high is the particle when its velocity is 50 feet per second? (e) at what time will the particle be 156 feet high? (f) how high is the particle at the end of $2\frac{1}{4}$ seconds? at the end of 4 seconds?

83. A bombing plane is traveling in a straight line at the rate of 160 miles per hour at an altitude of 3,600 feet. How far behind the target (measured horizontally) must the bomb be dropped so as to strike the target (neglecting atmospheric influences)?

84. Two automobiles are traveling toward each other on a straight road at the same uniform speed. If an airplane, flying at 96 miles per hour, passes over the second automobile 72 minutes after passing over the first and is 132 miles from the automobiles when they pass, find the speed of the automobiles.

85. A defense gunner on the ground wishes to hit the plane in Problem 83 by firing a shot straight up into the air with a velocity of 800 feet per second. How many seconds before the plane passes immediately above him must he fire the shot? How far away is the plane when the gun is fired (measured horizontally)? (Assume that the shot travels according to the law $S = V_0t - \frac{1}{2}gt^2$ and $g = 32$.)

86. A man is between two towns A and B and is twice as far from B as from A. If he travels 21 miles toward B he will then be $\frac{3}{5}$ as far from B as he is from A. Find the distance between A and B.

87. A road between A and B is 1 mile level, 1 mile uphill, 1 mile downhill, and 1 mile level. An automobile travels the first mile at a speed of 36 miles per hour, uphill at 20 miles per hour, and downhill at 45 miles per hour. At what speed must the automobile travel the last mile to average 40 miles per hour over the entire distance?

88. The safe load for a white pine beam supported at both ends is given by $W = \frac{500}{9}\frac{bd^2}{L}$, where b is breadth in inches, d thickness in inches, and L length in feet. What is the longest plank 10 inches wide and 2 inches thick that will hold with safety a man and wheelbarrow, if the man weighs 175 pounds and the loaded wheelbarrow 300 pounds? How long if two of these planks are used double? How long if a plank 10 by 4 inches is used? How thick must a 10-foot plank 10 inches wide be to hold the man and loaded wheelbarrow?

89. A rectangular flower bed a feet long and b feet wide has a rectangular border of turf of equal width. How wide is the border if the area of the flower bed and the border are equal?

90. The circumference of a rear wheel of a carriage exceeds the circumference of the front wheel by c feet. In traveling a distance of d miles the rear wheel makes n revolutions less than the front wheel. Find the circumference of each wheel.

91. A boat crew can row a distance of d miles downstream and return again in t hours. If the rate of the current of the stream is r miles per hour, find the rate of the boat in still water.

92. If a bicycle wheel c feet in circumference took t seconds less to revolve, the rate of the bicycle would be n miles more per hour. Find the rate of the bicycle.

93. Find the radius of a circle, the area of which would be doubled if the radius were increased by n feet.

94. A rectangular piece of tin is n times as long as it is wide. A square of side a is cut out from each corner, and the sides turned up so as to form a box of volume V. Find the dimensions of the piece of tin.

95. What value can be added either to the diameter d or to the height h of a cylinder so that the increase in volume will be the same in both cases?

96. An automobile radiator has a capacity of q quarts. If a quarts of water are drained out and replaced by a quarts of alcohol, and later another b quarts are drained out and replaced by alcohol, still later how much must be drained out and replaced by alcohol so that the final mixture will be p per cent alcohol? (Assume there is no loss by evaporation.)

97. It takes B b days longer to do a piece of work than it takes A to do the same work. How long will it take A alone to do the work if the two together can do the work in d days?

61. EQUATIONS IN THE QUADRATIC FORM

We may have equations that are not quadratic equations in the unknown, but *that are quadratic equations in some function of the unknown*. To solve such equations, we solve the quadratic for the function of the unknown. Having the value of the function, we can then solve for the unknown. To illustrate:

Example 1. Solve the equation $x^4 - 13x^2 + 36 = 0$.

Solution. This equation is not a quadratic in x, but is a quadratic in x^2 and may be written as

$$(x^2)^2 - 13x^2 + 36 = 0.$$

Factoring gives

$$(x^2 - 9)(x^2 - 4) = 0;$$

then

$$x^2 - 9 = 0 \quad \text{and} \quad x^2 - 4 = 0,$$

whence

$$x = \pm 3 \quad \text{and} \quad x = \pm 2.$$

If preferred, the function may be replaced by a second unknown. To illustrate:

Example 2. Solve $(x^2 + 3x)^2 - 2(x^2 + 3x) - 8 = 0$.

Solution. This is a quadratic equation in $x^2 + 3x$, and we may let $x^2 + 3x = y$. Then

$$y^2 - 2y - 8 = 0.$$

Factoring, we obtain

$$(y - 4)(y + 2) = 0;$$

solving for y gives

$$y = 4 \quad , \text{ or } \quad y = -2.$$

To solve for x, it is now necessary to solve the two equations

$$x^2 + 3x = 4 \quad \text{and} \quad x^2 + 3x = -2.$$

Factoring each equation, we get

$$(x + 4)(x - 1) = 0 \quad \text{and} \quad (x + 1)(x + 2) = 0;$$

solving for x gives

$$x = -4, 1 \quad \text{and} \quad x = -1, -2.$$

The equation $(x^2 + 3x)^2 - 2(x^2 + 3x) - 8 = 0$ can be solved by writing it as

$$[(x^2 + 3x) - 4][(x^2 + 3x) + 2] = 0;$$

then the solutions can be obtained by solving the two equations

$$x^2 + 3x - 4 = 0 \quad \text{and} \quad x^2 + 3x + 2 = 0.$$

Factoring each equation, we have

$$(x + 4)(x - 1) = 0 \quad \text{and} \quad (x + 1)(x + 2) = 0;$$

solving for x gives

$$x = -4, 1 \quad \text{and} \quad x = -1, -2.$$

Check: If $x = -4$, then $(16 - 12)^2 - 2(16 - 12) - 8 = 16 - 8 - 8 = 0$.

If $x = 1$, then $(1 + 3)^2 - 2(1 + 3) - 8 = 16 - 8 - 8 = 0$.

If $x = -1$, then $(1 - 3)^2 - 2(1 - 3) - 8 = 4 + 4 - 8 = 0$.

If $x = -2$, then $(4 - 6)^2 - 2(4 - 6) - 8 = 4 + 4 - 8 = 0$.

Another type of equation quite frequently met is one containing square roots, as: $\sqrt{2x - 1} + \sqrt{4x + 5} - \sqrt{19 - 3x} = 0$. These equations are called irrational equations.

To solve equations containing square roots, it is necessary in most cases to square both members of the equation to reduce the number of radicals.

To solve an equation containing but one square root, write the equation with the radical on one side of the equality sign and write all the terms free of the radical on the other side of the equality sign. Then, after squaring both members of the equation, solve the resulting equation, which is free of radicals.

To solve an equation containing two or three unlike square roots and more than two terms, write the equation with the most complicated radical on one side of the equality sign and the remaining terms on the other side of the equality sign. Then, after squaring both members of the equation, there will be but one radical remaining. The resulting equation can be solved as previously described.

Observe that $(\sqrt{a} + \sqrt{b})^2 = a + 2\sqrt{ab} + b$ and does not equal $a + b$.

Example 3. Solve $\sqrt{2x-1} + \sqrt{4x+5} - \sqrt{19-3x} = 0$,

Solution. We transpose one radical:

$$\sqrt{2x-1} + \sqrt{4x+5} = \sqrt{19-3x};$$

square both members:

$$2x - 1 + 2\sqrt{2x-1}\sqrt{4x+5} + 4x + 5 = 19 - 3x,$$

collect and transpose:

$$2\sqrt{8x^2+6x-5} = 15 - 9x,$$

square a second time:

$$32x^2 + 24x - 20 = 225 - 270x + 81x^2,$$

and collect and divide by 49:

$$x^2 - 6x + 5 = 0.$$

Then, solving for x, we obtain

$$x = 5 \quad \text{and} \quad x = 1.$$

Since both members of the original equation have been squared, these roots may be extraneous roots and hence may not be roots of the original equation. Checking in the original equation, we have:

$$\text{If } x = 5, \text{ then } \sqrt{9} + \sqrt{25} - \sqrt{4} \neq 0.$$
$$\text{If } x = 1, \text{ then } \sqrt{1} + \sqrt{9} - \sqrt{16} = 0.$$

Since $x = 1$ satisfies the equation, it is a root of the original equation, but $x = 5$ is not a root. Since $x = 5$ does not satisfy

the original equation, it is uncertain whether an error has been made in the process of solving the equation, or whether 5 is an extraneous root. By the following method we can determine whether the root is extraneous or not. *After squaring both members of any equation containing the square root, the roots of the derived equation must always be such that they will satisfy the original equation provided we can choose either sign for each radical.* To illustrate, $x = 5$ does not satisfy the given equation in Example 3, but $x = 5$ does satisfy the alternate equation

$$-\sqrt{2x-1}+\sqrt{4x+5}-\sqrt{19-3x}=0,$$

since

$$-\sqrt{9}+\sqrt{25}-\sqrt{4}=0.$$

This indicates that no error was made in the solution.

EXERCISES

Solve the following equations.

1. $\sqrt{x}-16=6.$
2. $\sqrt{x+9}=5.$
3. $\sqrt{x}=6-\dfrac{8}{\sqrt{x}}.$
4. $x-8\sqrt{x+9}+24=0.$
5. $x^{\frac{1}{4}}+6x^{\frac{1}{2}}-2=0.$
6. $4x^{\frac{1}{3}}-x^{\frac{2}{3}}-3=0.$
7. $3+\sqrt{8x-15}=2x.$
8. $x-5\sqrt{x}-36=0.$
9. $2\sqrt{x+1}=\sqrt{4-x}+\sqrt{x+6}.$
10. $2\sqrt{6-x}-\sqrt{2x-3}=3\sqrt{5-2x}.$
11. $\sqrt{2y+1}=5-\sqrt{y}.$
12. $\sqrt{x}-\sqrt{x+2}=3.$
13. $\sqrt{3x+3}=\sqrt{5x+6}-\sqrt{x-1}.$
14. $\sqrt{x-4}-\sqrt{2x-1}=\sqrt{3x+1}.$
15. $\sqrt{x-2}+\sqrt{x-5}-3=0.$
16. $\sqrt{2x+3}+\sqrt{7x+4}=\sqrt{x+5}.$
17. $a^3-7a^{\frac{3}{2}}-8=0.$
18. $8x^{-3}-65x^{-\frac{3}{2}}+8=0.$
19. $5x^{\frac{1}{3}}+2x^{\frac{2}{3}}-3=0.$
20. $\sqrt[3]{x^2}-5\sqrt[3]{x}-36=0.$
21. $(x-2)^2-2(x-2)-35=0.$
22. $(x^2-3x)^2-2(x^2-3x)=8.$
23. $m^{-5}-33m^{-\frac{5}{2}}+32=0.$

24. $\dfrac{x}{x+\sqrt{2x+6}}+\dfrac{x}{x-\sqrt{2x+6}}=-\dfrac{4}{3}$.

25. $x^4-29x^2+100=0$.

26. $(x^2+3x+1)^2=10x^2+30x+21$.

27. $x^2+x-8+\dfrac{12}{x^2+x}=0$.

28. $4(x-1)^{\frac{2}{3}}+36(x-1)^{-\frac{2}{3}}=25$.

29. $2x+4-6(x+2)^{-1}=-1$.

30. $\left(2x-\dfrac{3}{x}\right)^2+8x-\dfrac{12}{x}-5=0$.

62. PROPERTIES OF THE ROOTS OF A QUADRATIC EQUATION

The roots r_1 and r_2 of the quadratic equation $ax^2+bx+c=0$ are

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad r_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

If certain relations exist among the coefficients a, b, and c, then the roots of the quadratic equation will have certain properties, and conversely, if the roots of a quadratic equation have a given property, the coefficients of this equation must satisfy a given relation.

As previously stated, the expression b^2-4ac is called the discriminant of the equation $ax^2+bx+c=0$. If a, b, and c are real numbers, *then the value of the discriminant determines certain properties of the roots.*

PROPERTY (1). *If* b^2-4ac *is positive, the roots are real and unequal.*

PROPERTY (2). *If* b^2-4ac *is negative, the roots are imaginary.*

PROPERTY (3). *If* b^2-4ac *is zero, the roots are real and equal. The quadratic is a perfect square, and the roots are* $-\dfrac{b}{2a}$ and $-\dfrac{b}{2a}$.

For example, in the equation $x^2-6x+9=0$, $b^2-4ac=36-36=0$. Also, $x^2-6x+9=(x-3)^2=0$, and the roots are 3 and 3.

PROPERTY (4). *If* b^2-4ac *is zero, or is positive and a perfect square, the roots are rational. If* b^2-4ac *is positive and not a perfect square, the roots are irrational.* (For this case, a, b, and c must be rational numbers.)

PROPERTY (5). *If* $b = 0$, *and* a *and* c *have opposite signs, the roots are numerically equal but opposite in sign.*

If $b = 0$, the quadratic reduces to $ax^2 + c = 0$, and the roots are $\pm\sqrt{-\frac{c}{a}}$.

PROPERTY (6). *If* $c = 0$, *one of the roots is zero.*

If $c = 0$, the quadratic equation reduces to $ax^2 + bx = 0$. Factoring gives $x(ax + b) = 0$, and the roots are 0 and $-\frac{b}{a}$.

PROPERTY (7). *If* $b = c = 0$, *both roots are zero.*

If $b = c = 0$, the quadratic equation reduces to $ax^2 = 0$, and the roots are 0 and 0.

All the foregoing conditions are necessary and sufficient and hence their converses are true.

PROPERTY (8). *The sum of the two roots of a quadratic equation is equal to minus the coefficient of the unknown divided by the coefficient of the square of the unknown; that is, the sum of the two roots is equal to* $\frac{-b}{a}$.

The sum of the two roots is

$$r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a} = \frac{-b}{a}.$$

PROPERTY (9). *The product of the two roots of a quadratic equation is equal to the constant term divided by the coefficient of the square of the unknown; that is, the product of the two roots is equal to* $\frac{c}{a}$.

The product of the two roots is

$$r_1 \cdot r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}.$$

PROPERTY (10). *The quadratic equation having the roots* r_1 *and* r_2 *may be written in any one of the following three ways:* $(x - r_1)(x - r_2) = 0$; $x^2 - (r_1 + r_2)x + r_1r_2 = 0$; or $a(x - r_1)(x - r_2) = 0$.

In the following examples we shall illustrate a few of the foregoing properties by using the equation

$$kx^2 - 2kx + x - 2 = 0.$$

The standard form of this equation is

$$kx^2 + (-2k + 1)x - 2 = 0,$$

and the values of a, b, and c are

$$a = k, \qquad b = -2k + 1, \qquad \text{and} \qquad c = -2.$$

Example 1. Determine k so that the two roots of $kx^2 - 2kx + x - 2 = 0$ are equal.

Solution. From Property (3), the roots will be equal if

$$b^2 - 4ac = 0;$$

hence,

$$(-2k + 1)^2 - 4k(-2) = 0.$$

Simplifying, we have

$$4k^2 + 4k + 1 = 0;$$

factoring gives

$$(2k + 1)^2 = 0;$$

solving for k, we obtain

$$k = -\tfrac{1}{2}.$$

Check: If $k = -\frac{1}{2}$, the original equation reduces to $x^2 - 4x + 4 = (x - 2)^2 = 0$, hence, the two roots are 2 and 2.

Example 2. Determine k so that the two roots of $kx^2 - 2kx + x - 2 = 0$ are numerically equal but opposite in sign.

Solution. From Property (5), the roots are numerically equal but opposite in sign if

$$b = 0;$$

hence,

$$-2k + 1 = 0.$$

Solving for k gives

$$k = \tfrac{1}{2}.$$

Check: If $k = \frac{1}{2}$, the original equation reduces to $x^2 - 4 = 0$, and the two roots are 2 and -2.

Example 3. Determine k so that the sum of the two roots of $kx^2 - 2kx + x - 2 = 0$ is 1.

Solution. From Property (8), the sum of the roots is $\frac{-b}{a}$, hence,

$$-\frac{b}{a} = 1.$$

Therefore,

$$-\frac{-2k+1}{k}=1.$$

Solving for k gives

$$k=1.$$

Check: If $k=1$, the original equation reduces to $x^2-x-2=0$, and the roots are 2 and -1. Thus, the sum of the roots is 1.

Example 4. Determine k so that 3 is a root of $kx^2-2kx+x-2=0$.

Solution. If 3 is to be a root it must satisfy the equation. Thus, substituting 3 for x in the original equation, we have

$$9k-6k+3-2=0;$$

collecting, we have

$$3k+1=0;$$

solving for k gives

$$k=-\tfrac{1}{3}.$$

Check: If $k=-\frac{1}{3}$, the original equation reduces to $x^2-5x+6=0$, and the roots are 3 and 2.

Example 5. Find the quadratic equation whose roots are -2 and $\frac{3}{2}$.

Solution. From Property (10) this equation is either

$$(x+2)(x-\tfrac{3}{2})=0; \qquad \text{or} \qquad 2(x+2)(x-\tfrac{3}{2})=0.$$

either form is equivalent to

$$(x+2)(2x-3)=0,$$

or

$$2x^2+x-6=0.$$

Example 6. Find the quadratic equation whose roots are $3\pm\sqrt{2i}$.

Solution. Using the second form in Property (10), we see that this equation might be written as

$$x^2-(r_1+r_2)x+r_1r_2=0.$$

The sum of the roots is

$$r_1+r_2=(3+\sqrt{2i})+(3-\sqrt{2i})=6,$$

and the product of the roots is

$$\begin{aligned} r_1r_2 &= (3+\sqrt{2i})(3-\sqrt{2i}) \\ &= 9-2i^2 \\ &= 11. \end{aligned}$$

Therefore the equation is

$$x^2-6x+11=0.$$

Check: As an exercise, the student should solve this equation for the two roots.

63. INEQUALITIES

If we wish to determine whether the roots of a quadratic equation are real or imaginary, it is necessary to determine whether the discriminant ($b^2 - 4ac$) is equal to or greater than zero, or is less than zero. The inequality symbol $>$ is used to denote greater than, and the inequality symbol $<$ for less than, and we shall restrict their use to real numbers.

We can operate with the inequality sign in the same manner as with the equality sign, except for the following three operations. (1) *Multiplying both members of an inequality by a negative number,* or (2) *taking the reciprocal of both members of the inequality, reverses the inequality.* (3) *Taking an even power of both members of an inequality may reverse the inequality, if one or both members are negative.* Thus, if

$$a > b$$

then

$$a + c > b + c,$$

$$a - c > b - c,$$

$$ac > bc, \qquad (c \text{ positive})$$

$$\frac{a}{c} > \frac{b}{c}, \qquad (c \text{ positive})$$

$$\sqrt[n]{a} > \sqrt[n]{b},$$

and

$$a^n > b^n; \qquad (a \text{ and } b \text{ positive})$$

but

$$-a < -b,$$

and

$$\frac{1}{a} < \frac{1}{b}.$$

To illustrate the last two conditions, we have that $5 > 2$, but

$$-5 < -2 \qquad \text{and} \qquad \frac{1}{5} < \frac{1}{2}.$$

Illustrating the third condition

$$-5 < -2 \qquad \text{but} \qquad (-5)^2 > (-2)^2.$$

Example 1. If a and b are not equal, prove that $a^2 + b^2 > 2ab$.

Solution. Subtracting $2ab$ from both members of the inequality, we have

$$a^2 - 2ab + b^2 > 0,$$

or

$$(a - b)^2 > 0,$$

which is always a true statement since the square of any real number is a positive number; that is, it is a number greater than zero.

Example 2. For what values of k are the roots of $x^2 - kx + 9 = 0$ real?

Solution. If the roots are to be real, the discriminant must be zero or positive; that is, it is necessary that

$$b^2 - 4ac \geqq 0.$$

Thus it is necessary that

$$k^2 - 36 \geqq 0,$$

or

$$(k + 6)(k - 6) \geqq 0.$$

The product $(k + 6)(k - 6)$ will be zero if $k = \pm 6$, and $(k + 6)(k - 6)$ will be positive if $k + 6$ and $k - 6$ are both negative or both positive. Hence the roots of $x^2 - kx + 9 = 0$ are real if $k \leqq -6$, or if $k \geqq 6$.

QUESTIONS

1. How many roots does a quadratic equation have?
2. When are the roots of a quadratic equation equal?
3. When are the roots of a quadratic equation rational?
4. When are the roots of a quadratic equation irrational?
5. When are the roots of a quadratic equation real?
6. When are the roots of a quadratic equation imaginary?
7. When are the roots of a quadratic equation numerically equal but opposite in sign?
8. When will one root of a quadratic equation be zero?
9. What is the sum of the roots of a quadratic equation?
10. What is the product of the roots of a quadratic equation?
11. How does one determine whether a given number is a root of a given equation?
12. How do we form a quadratic equation when its roots are given?

EXERCISES

Give the sum and product of the roots for each of the following twelve equations.

1. $20x^2 - 13x + 240 = 0$.
2. $3x^2 - 5x + 4 = 0$.
3. $x^2 - 65x - 350 = 0$.
4. $10x^2 - 25x = 12$.
5. $3x^2 + 8x = 50$.
6. $x^2 = 6x - 2$.
7. $3x^2 + x = 3x + 1$.
8. $kx^2 + 2x - 5 = 0$.
9. $kx^2 + kx = 7$.
10. $x^2 - 2kx + k = 5$.
11. $x^2 - 3x + 4 = kx + 2k$.
12. $5 - ax + 3x^2 = b + 2x - cx^2$.

Are the roots of each of the following equations (Exercises 13 through 24) real or imaginary?

13. $2x^2 - 10x + 9 = 0$.
14. $3x^2 - 5x + 4 = 0$.
15. $x^2 = 9x - 20$.
16. $10x^2 - 25x = 12$.
17. $13x^2 - 625x - 422 = 0$.
18. $x^2 = 6x - 2$.
19. $3x^2 + x = 3x + 1$.
20. $2 = 5x - 3x^2$.
21. $4 - 2x + 6x^2 = 0$.
22. $x^2 + 10x = 10$.
23. $6x^2 - 2x + 5 = 0$.
24. $4x^2 = 12x - 9$.

For what values of k will the roots of the equations in Exercises 25 through 39 be equal; be real; be imaginary?

25. $x^2 + kx + 9 = 0$.
26. $kx^2 + 2x - 5 = 0$.
27. $kx^2 - 2x + 5 = 0$.
28. $kx^2 + 2kx - 2 = 0$.
29. $x^2 - kx + k + 3 = 0$.
30. $2x^2 - 3kx + 18 = 0$.
31. $x^2 - 8x + k = 0$.
32. $3x^2 - 6x + k = 0$.
33. $2x^2 - 3x + 2k - 1 = 0$.
34. $kx^2 - 4x - k + 4 = 0$.
35. $x^2 + kx - 4 = 0$.
36. $x^2 - kx - k + 3 = 0$.
37. $x^2 + 2x + 1 = kx^2 - kx + 4$.
38. $3x^2 + 2kx = 3$.
39. $9x^2 + 7x + k^2 = 0$.

In each of the following equations (Exercises 40 through 55) determine k so that the indicated conditions will be satisfied.

40. $x^2 - 3x + k = 0$; the product of the roots is 6.
41. $x^2 - 5x + k = 0$; the product of the roots is 5.
42. $x^2 - kx + x - 5 = 0$; the sum of the roots is -4.
43. $kx^2 - 4x + 2k = 0$; the sum of the roots is twice the product of the roots.
44. $kx^2 - 5x + 2k = 0$; 3 is a root.
45. $kx^2 + 2x - 5k = kx$; 3 is a root.
46. $10x^2 + 6kx - 5k + 8 = 0$; one root is zero.
47. $x^2 - kx + 5 = kx^2 - x + 5k$; one root is zero.
48. $kx^2 - 3kx + 2k + 4x - 4 = 0$; the roots are equal.
49. $x^2 - 5x + k = 0$; the roots are equal.
50. $(3x + k)^2 = 9$; the sum of the roots is 6.
51. $x^2 - x - 1 = kx^2 + kx + 1$; the roots are equal.

52. $x^2 - 3x + 4 + kx - 3k = 0$; the roots are real.

53. $k - x - x^2 = 0$; the roots are imaginary.

54. $kx^2 + kx + k + x^2 + 1 = 0$; the product of the roots is 5.

55. $6x^2 - 5kx - 2k - x + 4 = 0$; the roots are arithmetically equal but opposite in sign.

Find the quadratic equation with integral coefficients whose roots are given in Exercises 56 through 70.

56. 2, 3.

57. -3, 1.

58. -3, -5.

59. 2, -5.

60. $\frac{2}{3}$, $\frac{3}{4}$.

61. $\frac{a}{b}$, $\frac{b}{a}$.

62. i, $-i$.

63. a, $\frac{1}{a}$.

64. $1 + \sqrt{3}$, $1 - \sqrt{3}$.

65. $\frac{3}{4}$, $-\frac{4}{5}$.

66. $-3 + 2\sqrt{2}$, $-3 - 2\sqrt{2}$.

67. $\frac{1}{2} + \frac{1}{2}\sqrt{3}$, $\frac{1}{2} - \frac{1}{2}\sqrt{3}$.

68. $-2 + i$, $-2 - i$.

69. $3 + i$, $3 - i$.

70. $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

71. If r_1 and r_2 are the roots of $ax^2 + bx + c = 0$, form the equations whose roots are:

(a) $-r_1$, $-r_2$;

(b) $2r_1$, $2r_2$;

(c) r_1^2, r_2^2;

(d) $\frac{1}{r_1}$, $\frac{1}{r_2}$.

72. If r_1 and r_2 are the roots of $ax^2 + bx + c = 0$, find the value of the following in terms of a, b, and c.

(a) $r_1^2 + r_2^2$;

(b) $\frac{1}{r_1} + \frac{1}{r_2}$;

(c) $\frac{r_1}{r_2} + \frac{r_2}{r_1}$;

(d) $(r_1 - r_2)^2$.

For what real values of x do the inequalities expressed in Exercises 73 through 88 hold?

73. $6x - 5 > 7$.

74. $6x - 3 > 2$.

75. $x^2 < 9$.

76. $2x + 1 < 4$.

77. $3x^2 + 7x < 6$.

78. $3x^2 - x - 10 < 0$.

79. $\frac{3x + 2}{x - 3} > 0$.

80. $\frac{2x - 3}{3x - 2} < 0$.

81. $x^2 > 9$.

82. $x + \frac{1}{x} > 2$.

83. $\frac{2}{x - 3} < \frac{3}{x - 2}$.

84. $\frac{x + 5}{2x - 3} > 0$.

85. $\dfrac{2x+5}{3} > \dfrac{3-x}{-3}$.

86. $(x-3)(x+1) > x(x-1)$.

87. $(x-2)(x+4) < (x+1)(x-3)$.

88. $\dfrac{x+4}{x-2} > \dfrac{x-8}{x-6}$.

Prove that the following inequalities are true if a and b are distinct positive numbers.

89. $\dfrac{a}{b} + \dfrac{b}{a} > 2$.

90. $\dfrac{a+b}{2} > \sqrt{ab}$.

91. $\dfrac{a+b}{2a} > \dfrac{2b}{a+b}$.

92. $\dfrac{a+b}{2} > \dfrac{2ab}{a+b}$.

93. $a + \dfrac{b^2}{a} > 2b$.

94. $a^3 + b^3 > a^2b + ab^2$.

CHAPTER XII

Theory of Equations

64. HIGHER-DEGREE EQUATIONS

In 1545, Cardan published a partial algebraic solution of the general cubic equation and the general quartic equation; that is, he gave a formula for the solution of the general cubic and the general quartic in terms of radicals. However, the Italian Ferro (1465–1526), in about 1515, was the first to find an algebraic solution of certain cubics, and another Italian, Ferrari (1522–1565), while yet a youth, was the first to find an algebraic solution of the quartic, but these solutions were first published by Cardan. Ferro solved the equation $x^3 + px = q$, and Cardan observed how to transform the general cubic to this form. Not having a clear idea of the three cube roots of unity, nor the four fourth roots of unity, they failed to incorporate all the roots in their general solution. However, Cardan did give all the roots for several numerical equations that he had solved, which showed that he had some idea of how many roots the cubic equation should have. It was not until 1824 that the Norwegian mathematician N. H. Abel (1802–1829), proved that the general fifth-degree equation has no algebraic solution; that is, no algebraic formula can be found that will give the roots of the general quintic equation. By means of the theory of substitution groups, it can be shown that the general equation of degree greater than four cannot have an algebraic solution; that is, the roots of the general algebraic equation of degree greater than four cannot be represented in terms of radicals.

The linear, quadratic, cubic, and quartic equations have each their special methods for obtaining their solutions, but these methods are not sufficient to obtain the solutions of equations of higher degree. Therefore, it is necessary to develop methods whereby the roots of algebraic equations of higher degree can be found or approximated. These methods of solution will be independent of the degree of the equation and hence can be used

to solve equations of any degree. The formulas for the solutions of the cubic and quartic are given in Articles 78, 79, and 80, but since these methods are rather complicated, we shall solve these equations by the methods used to solve the general equation.

A proof of the theorem that every algebraic equation with real or imaginary coefficients must have at least one root, real or imaginary, was not given until 1797. In this year the German mathematician K. F. Gauss (1777–1855) gave in his Dissertation a proof of the **fundamental theorem of algebra,** which states that *every algebraic equation must have at least one root, real or imaginary.* This was published in 1799.

In this chapter we shall study only those algebraic equations that have real coefficients, and we shall consider the roots of such equations under the three headings, (1) rational roots, (2) imaginary roots, and (3) irrational roots.

65. RATIONAL ROOTS

If an equation has a rational root, this root must be of the form $\frac{p}{q}$. We may suppose that p and q do not have a common factor greater than 1; that is, we shall assume that the fraction $\frac{p}{q}$ is in its reduced form. If $\frac{p}{q}$ is a root of the algebraic equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n = 0, \qquad (1)$$

then

$$f\left(\frac{p}{q}\right) = a_0\left(\frac{p}{q}\right)^n + a_1\left(\frac{p}{q}\right)^{n-1} + a_2\left(\frac{p}{q}\right)^{n-2} + \cdots + a_{n-2}\left(\frac{p}{q}\right)^2 + a_{n-1}\frac{p}{q} + a_n = 0. \qquad (2)$$

Multiplying each member of Equation (2) by q^{n-1}, we have

$$a_0\frac{p^n}{q} + a_1p^{n-1} + a_2p^{n-2}q + \cdots + a_{n-2}p^2q^{n-3} + a_{n-1}pq^{n-2} + a_nq^{n-1} = 0. \qquad (3)$$

If all the coefficients in Equation (1) are integers, then the sum of all the terms in Equation (3) after the first term is an integer. Therefore the first term $a_0\frac{p^n}{q}$ must be an integer if the sum of all

the terms is zero. There are two conclusions we can draw from this fact:

First, if $a_0 = 1$, then $a_0 \frac{p^n}{q} = \frac{p^n}{q}$ is an integer only if q equals 1. Therefore, *the rational roots of an algebraic equation with integral coefficients must be integers whenever* $a_0 = 1$.

Second, if a_0 is different from 1, then $a_0 \frac{p^n}{q}$ is an integer only if a_0 contains q as a factor. That is, *the rational roots of an algebraic equation with integral coefficients must be either integers or fractions whose denominators are factors of* a_0. (Observe that a_0 is the coefficient of the highest power of x.)

Next, multiply each term of Equation (2) by $\frac{q^n}{p}$; then

$$a_0 p^{n-1} + a_1 p^{n-2} q + a_2 p^{n-3} q^2 + \cdots + a_{n-2} p q^{n-2} + a_{n-1} q^{n-1} + a_n \frac{q^n}{p} = 0. \qquad (4)$$

If all the coefficients a_i $(i = 0, 1, 2, \cdots, n)$ are integers, then the sum of all the terms in Equation (4) up to the last term is an integer. Therefore the last term $a_n \frac{q^n}{p}$ must be an integer, if the sum of all the terms is zero. Now $a_n \frac{q^n}{p}$ is an integer only if a_n contains p as a factor. Therefore, *the numerator* p *of any rational root of an algebraic equation with integral coefficients must be a factor of* a_n, *the constant term.*

These results may be stated as the following theorems.

Theorems on Rational Roots

THEOREM 1. *If an algebraic equation with integral coefficients, the coefficient of the highest power of* x *being* 1, *has any rational roots, they must be* **integers** *and factors of the constant term* (the term free of x).

THEOREM 2. *If an algebraic equation with integral coefficients, the coefficient of the highest power of* x *being different from* 1, *has any rational roots, they are either integers that are factors of the constant term or fractions whose numerators are factors of the constant term and whose denominators are factors of the coefficient of the highest power of* x. To illustrate:

By Theorem 1, the rational roots of

$$x^3 + 4x^2 + x - 6 = 0$$

must be included in the following set of numbers: ± 1, ± 2, ± 3, ± 6. Since $a_0 = 1$, all the rational roots of $x^3 + 4x^2 + x - 6 = 0$ must be integers and must be factors of 6.

By Theorem 2, the rational roots of

$$6x^3 - 5x^2 - 19x + 10 = 0$$

must be included in the following set of numbers; ± 1, ± 2, ± 5, ± 10, $\pm \frac{1}{2}$, $\pm \frac{2}{2}$, $\pm \frac{5}{2}$, $\pm \frac{10}{2}$; $\pm \frac{1}{3}$, $\pm \frac{2}{3}$, $\pm \frac{5}{3}$, $\pm \frac{10}{3}$; $\pm \frac{1}{6}$, $\pm \frac{2}{6}$, $\pm \frac{5}{6}$, $\pm \frac{10}{6}$. (Some of these numbers are equal, but the complete list is given to show the general method.) All the rational roots of this equation must be such that the numerators are factors of 10 and the denominators are factors of 6.

To find the rational roots of a given equation, it is necessary to substitute in the equation those numbers that might be roots and see whether or not they satisfy the given equation. To test each possible number by substitution may be long and difficult, unless we can derive methods that will shorten and simplify the work. In the remainder of this chapter we shall derive such methods and prove many useful theorems.

66. REMAINDER THEOREM

Observe that

$$\textbf{dividend} = \textbf{quotient} \times \textbf{divisor} + \textbf{remainder}. \tag{5}$$

To illustrate, if we divide 125 by 7, the quotient is 17 and the remainder is 6, and we may write

$$125 = 17 \cdot 7 + 6.$$

Likewise, if we divide $f(x)$, a polynomial in x, by $x - r$ and if we denote the quotient by the polynomial $Q(x)$ and the remainder, which is free of x, by R, then we may write

$$f(x) = Q(x)(x - r) + R, \tag{6}$$

which is an identity and hence is true for all values of x. Substituting r for x, we have

$$f(r) = Q(r)(0) + R, \tag{7}$$

hence,

$$R = f(r). \tag{8}$$

Therefore,

$$f(x) = Q(x)(x - r) + f(r). \tag{9}$$

This result may be stated as the following theorem.

THE REMAINDER THEOREM. *If a polynomial $f(x)$ be divided by $x - r$ until a remainder free of x is obtained, then this remainder is equal to $f(r)$,* the value of $f(x)$ when r is substituted for x.

To illustrate, suppose we divide $f(x) = x^3 + 4x^2 + x - 6$ by $x - 2$; then

$$\begin{array}{r|l} x^3 + 4x^2 + x - 6 & x - 2 \\ \underline{x^3 - 2x^2} & \underline{x^2 + 6x + 13} \\ 6x^2 + x & \\ \underline{6x^2 - 12x} & \\ 13x - 6 & \\ \underline{13x - 26} & \\ 20 & \end{array}$$

Thus, dividing $x^3 + 4x^2 + x - 6$ by $x - 2$, the quotient is $x^2 + 6x + 13$ and the remainder is 20. The remainder 20 is the value of $f(2)$. This can be verified by substituting 2 for x in $f(x)$, for then,

$$\begin{aligned} f(2) &= (2)^3 + 4(2)^2 + (2) - 6 \\ &= 8 + 16 + 2 - 6 \\ &= 20. \end{aligned}$$

67. THE FACTOR THEOREM

If r is a root of the equation

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0,$$

then $x - r$ is a factor of $f(x)$.

By Equation (9)

$$f(x) = Q(x)(x - r) + f(r),$$

and by hypothesis

$$f(r) = 0,$$

since r is a root. Hence it follows that

$$f(x) = Q(x)(x - r). \tag{10}$$

To illustrate: -2 is a root of the equation $x^3 + 4x^2 + x - 6 = -8 + 16 - 2 - 6 = 0$ because $f(-2) = (-2)^3 + 4(-2)^2 + (-2) - 6$

$16 - 2 - 6 = 0$ satisfies the equation. Then by the factor theorem, $x + 2$ should be a factor of $f(x) = x^3 + 4x^2 + x - 6$. [$x - r = x - (-2) = x + 2$.] Dividing, as follows;

$$\begin{array}{r|l} x^3 + 4x^2 + x - 6 & x + 2 \\ \underline{x^3 + 2x^2} & \underline{x^2 + 2x - 3} \\ 2x^2 + x & \\ \underline{2x^2 + 4x} & \\ -3x - 6 & \\ \underline{-3x - 6} & \end{array}$$

we see that $f(x)$ does contain the factor $x + 2$. The factors of $x^3 + 4x^2 + x - 6$ are, therefore,

$$\begin{aligned} f(x) = x^3 + 4x^2 + x - 6 &= (x + 2)(x^2 + 2x - 3) \\ &= (x + 2)(x + 3)(x - 1). \end{aligned}$$

68. THE NUMBER OF ROOTS OF AN EQUATION

Every algebraic equation of degree n *has* n *roots, real or imaginary, and no more.* (The n roots need not be all distinct.)

By the fundamental theorem of algebra, every algebraic equation $f(x) = 0$ has at least one root, say r_1. Then, by the factor theorem, $f(x)$ contains the factor $(x - r_1)$. Hence, we may write

$$f(x) = (x - r_1)Q_1(x). \tag{11}$$

If $f(x)$ is a polynomial of degree n, $Q_1(x)$ is a polynomial of degree $n - 1$.

Since $Q_1(x) = 0$ is an algebraic equation, then by the fundamental theorem of algebra, this equation has at least one root, say r_2, and by the factor theorem, $Q_1(x)$ contains the factor $(x - r_2)$. Hence, we may write

$$Q_1(x) = (x - r_2)Q_2(x). \tag{12}$$

By Equation (11)

$$f(x) = (x - r_1)Q_1(x); \tag{13}$$

then by Equation (12)

$$f(x) = (x - r_1)(x - r_2)Q_2(x), \tag{14}$$

where $Q_2(x)$ is a polynomial of degree $n - 2$.

This process can be continued until the final quotient is a constant. Since the kth quotient $Q_k(x)$ is of degree $n - k$, it

follows that the nth quotient $Q_n(x)$ is of degree zero and hence is a constant. Thus, *every polynomial of degree* $\boldsymbol{n}$ *must have at least* $\boldsymbol{n}$ *linear factors, and we may write*

$$f(x) = (x - r_1)(x - r_2)(x - r_3) \cdots (x - r_{n-1})(x - r_n)Q_n, \tag{15}$$

where Q_n is a constant. Since this product must be equal to the original polynomial $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, which has a_0 for the coefficient of x^n, it follows that Q_n must equal a_0. Hence we may write

$$f(x) = a_0(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_{n-1})(x - r_n) = 0, \tag{16}$$

and thus the given algebraic equation $f(x) = 0$ of degree n has the n roots $r_1, r_2, r_3, \cdots, r_n$.

We have just proved that if an algebraic equation of degree n has one root, it must have at least n roots. We shall now show that it cannot have more than n roots.

If $f(x) = 0$ has an $(n + 1)$st root, say r, different from the n roots, $r_1, r_2, r_3, \cdots, r_{n-1}, r_n$, then the product

$$f(r) = a_0(r - r_1)(r - r_2)(r - r_3) \cdots (r - r_{n-1})(r - r_n) \tag{17}$$

must be zero. But this product can be zero only if one of its factors is zero. Since r was assumed to be different from $r_1, r_2, r_3, \cdots, r_n$, it follows that none of these factors can be zero. Hence $f(r)$ cannot be zero and r cannot be a root of $f(x) = 0$. This completes the proof that every algebraic equation of degree n must have n roots and can have no more than n roots. It follows from this that *it is possible to write the polynomial* $\boldsymbol{f(x)}$ *of degree* $\boldsymbol{n}$ *as the product of* $\boldsymbol{n}$ *linear factors in but one way, and the roots of the equation* $\boldsymbol{f(x) = 0}$ *are the zeros of these* $\boldsymbol{n}$ *linear factors.*

QUESTIONS

1. What is the fundamental theorem of algebra?
2. What are the two theorems on rational roots?
3. What is the remainder theorem?
4. What is the factor theorem?
5. How many roots does an algebraic equation have?

EXERCISES

1. Give the set of numbers which includes all the rational roots of each of the following equations,

(a) $x^3 - 5x^2 - 4x + 20 = 0$; (b) $x^3 - 5x^2 - 2x + 24 = 0$;
(c) $x^3 + 6x^2 + 5x - 12 = 0$; (d) $2x^3 - 3x^2 - 3x + 2 = 0$;
(e) $2x^3 + 7x^2 - 7x - 12 = 0$; (f) $6x^3 + 19x^2 + x - 6 = 0$.

2. What is the remainder when $x^3 - x + 1$ is divided by $x - 1$?
3. What is the remainder when $x^n - a^n$ is divided by $x - a$?
4. What is the remainder when $x^n - a^n$ is divided by $x + a$?
5. What is the remainder when $x^n + a^n$ is divided by $x - a$?
6. What is the remainder when $x^n + a^n$ is divided by $x + a$?
7. Does $x^n - a^n$ contain the factor $x - a$?
8. For what values of n will $x^n - a^n$ contain the factor $x + a$?
9. Does $x^n + a^n$ contain the factor $x - a$?
10. For what values of n will $x^n + a^n$ contain the factor $x + a$?
11. What value must k have so that $x^3 + 2kx^2 - x + 5$ contains the factor $x - 1$?
12. Solve each of the following equations by factoring:

(a) $x^3 - 2x^2 - 5x + 6 = 0$;
(b) $x^4 - 5x^2 + 4 = 0$;
(c) $x^3 - 2x^2 - x + 2 = 0$.

69. SYNTHETIC DIVISION AND SUBSTITUTION

Suppose we wish to find the linear factors of

$$f(x) = x^3 + 4x^2 + x - 6.$$

From the factor theorem and the theorem concerning rational roots, it follows that all the possible rational factors of $x^3 + 4x^2 + x - 6$ must be included in the set $(x \pm 1)$, $(x \pm 2)$, $(x \pm 3)$, $(x \pm 6)$.

In Article 66 we divided $x^3 + 4x^2 + x - 6$ by $x - 2$ by long division and found the quotient to be $x^2 + 6x + 13$ and the remainder to be 20; that is,

$$x^3 + 4x^2 + x - 6 = (x - 2)(x^2 + 6x + 13) + 20.$$

Instead of using long division it is possible to abbreviate the work as shown below. This abbreviated method for division is called **synthetic division.** Both methods of division are shown below for the purpose of comparison.

By synthetic division we have

$$\begin{array}{r|l} 1+4+\ \ 1-\ \ 6 & 2 \\ \quad +2+12+26 & \\ \hline 1+6+13+20 = f(2) & \end{array}$$

By long division we have

$$\begin{array}{rl|l} x^3+4x^2+\ \ x-\ \ 6 & & x-2 \\ \cline{3-3} x^3-2x^2 & & x^2+6x+13 \\ \cline{1-1} \quad 6x^2+\ \ x & & \\ \quad 6x^2-12x & & \\ \cline{1-1} \quad 13x-\ \ 6 & & \\ \quad 13x-26 & & \\ \cline{1-1} \quad 20 & & \end{array}$$

In synthetic division when dividing $f(x)$ by $x-r$, we write only the coefficients of $f(x)$, and use r instead of $x-r$.

In long division the coefficient of x^2, which is the first term in the quotient, is 1, so in synthetic division we write the first coefficient 1 immediately below the line. In long division -2 is multiplied by x^2 and the product $-2x^2$ is subtracted from the second term $4x^2$. In synthetic division we multiply $+2$ by 1 and therefore must **add** the product 2 to the second term 4. Continue by multiplying the sum 6 by 2 and adding the product 12 to the third term 1, multiply the sum 13 by 2 and add the product 26 to the fourth term -6. This last sum 20 is the remainder and the other sums 1, 6, and 13 are the coefficients of the quotient. The quotient is of one degree less than $f(x)$. Hence, if x^3+4x^2+x-6 is divided by $x-2$ the quotient is $x^2+6x+13$ and the remainder is 20, and by the remainder theorem, 20 is the value of $f(2)$.

From the foregoing illustration we see that synthetic division is much shorter than long division, and that synthetic division is a very simple method for obtaining the values of $f(x)$ for different values of x.

This abbreviated way of representing long division is called **synthetic division,** *and this abbreviated method for finding the value of* $f(r)$ *is called* **synthetic substitution.**

To divide $f(x)$ *by* $x-r$ *by synthetic division, or to find the value of* $f(r)$ *by synthetic substitution: first, write the coefficients of* $f(x)$ *in*

the successive order of the descending powers of x (***supplying*** **0** ***for all the missing terms***)*; second, after drawing the horizontal line below these coefficients, leaving space for other figures, write the first term, which is the coefficient of the highest power of* x*, immediately below the horizontal line. Then multiply this term by* r*, write the product above the line under the second term, and add the product to the second term, writing the sum below the line; then multiply this sum by* r*, add the product to the third term, and so on. The successive sums below the line are the coefficients of the quotient, which is one degree lower in* x *than the degree of* $f(x)$*, and the last sum is the remainder and is the value of* $f(r)$.

It should be observed that if we divide $f(x)$ by $x + 2$ by synthetic division, $x - r = x - (-2) = x + 2$. To illustrate:

Example 1. Factor $x^3 + 4x^2 + x - 6$.

Solution. The work by synthetic division is as follows. Testing $(x + 2)$, $r = -2$ and we have

$$\begin{array}{rrrrl} 1 & +4 & +1 & -6 & \lfloor -2 \\ & -2 & -4 & +6 & \\ \hline 1 & +2 & -3 & +0 & \end{array} = f(-2)$$

Since the remainder is 0, $f(-2) = 0$, $x + 2$ is a factor of $x^3 + 4x^2 + x - 6$, and the quotient is $x^2 + 2x - 3$. Thus, we may write

$$x^3 + 4x^2 + x - 6 = (x + 2)(x^2 + 2x - 3).$$

To find the remaining factors of $x^3 + 4x^2 + x - 6$, we need only to factor $x^2 + 2x - 3$. Since these factors are $(x + 3)$ and $(x - 1)$, it follows that

$$x^3 + 4x^2 + x - 6 = (x + 2)(x + 3)(x - 1).$$

When a root r of the equation $f(x) = 0$ has been found, it is possible to write $f(x)$, thus:

$$f(x) = (x - r)Q(x) = 0.$$

The remaining roots of $f(x) = 0$ can always be found by finding the roots of the equation $Q(x) = 0$, which is called the **depressed equation.** To illustrate:

Example 2. Find all the rational roots of $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$.

Solution. All the rational roots of this equation must be included in the set of numbers ± 1, ± 2, ± 3, ± 4, ± 6, ± 12.

Testing 1, we have

$$\begin{array}{r|l} 1+2-7-\ \ 8+12 & \underline{1} \\ \quad +1+3-\ \ 4-12 & \\ \hline 1+3-4-12+\ \ 0 & \end{array}$$

Therefore 1 is a root.

Testing 2 in the depressed equation $x^3 + 3x^2 - 4x - 12 = 0$, we have

$$\begin{array}{r|l} 1+3-\ \ 4-12 & \underline{2} \\ \quad +2+10+12 & \\ \hline 1+5+\ \ 6+\ \ 0 & \end{array}$$

Therefore 2 is a root.

Solving the second depressed equation $x^2 + 5x + 6 = 0$, we have

$$x^2 + 5x + 6 = (x + 2)(x + 3) = 0,$$

and the roots of this equation are -2 and -3.

Thus the roots of $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$ are 1, 2, -2, and -3.

It is possible, however, if all the roots are real and distinct, to find all the roots of $f(x) = 0$ by using the coefficients of the original equation when testing for each root.

If $f(x)$ is divisible by $(x - r)^m$ but not by $(x - r)^{m+1}$, then r is called a **root of multiplicity** m of the equation $f(x) = 0$, or is sometimes called a **multiple root of order** m. A root of multiplicity 2 is called a **double root,** a root of multiplicity 3 is called a **triple root,** and so on.

To determine whether or not r is a multiple root, it is necessary to see whether or not r is a root of the first depressed equation $Q(x) = 0$, where $Q(x)$ is the quotient obtained by dividing $f(x)$ by $(x - r)$ by synthetic division.

A root r is a multiple root of order m if r is a root both of the original equation and of the first $(m - 1)$ depressed equations, but is not a root of the mth depressed equation.

We shall now illustrate how to solve an equation that has multiple roots.

Example 3. Solve $x^5 - 8x^4 + 21x^3 - 14x^2 - 20x + 24 = 0$.

Solution. Testing 2, we have

$$\begin{array}{r|l} 1-8+21-14-20+24 & \underline{2} \\ \quad +2-12+18+\ \ 8-24 & \\ \hline 1-6+\ \ 9+\ \ 4-12+0 & \end{array}$$

Testing 2 in the first depressed equation $x^4 - 6x^3 + 9x^2 + 4x - 12 = 0$, we have

$$\begin{array}{r|l} 1 - 6 + 9 + 4 - 12 & \lfloor 2 \\ \quad +2 - 8 + 2 + 12 & \\ \hline 1 - 4 + 1 + 6 + 0 & \end{array}$$

Testing 2 in the second depressed equation $x^3 - 4x^2 + x + 6 = 0$, we have

$$\begin{array}{r|l} 1 - 4 + 1 + 6 & \lfloor 2 \\ \quad +2 - 4 - 6 & \\ \hline 1 - 2 - 3 + 0 & \end{array}$$

Hence 2 is a triple root.

Solving the third depressed equation $x^2 - 2x - 3 = 0$, we have

$$x^2 - 2x - 3 = (x - 3)(x + 1) = 0.$$

Hence

$$x = 3 \quad \text{and} \quad x = -1.$$

Therefore, the five roots of the given equation are 2, 2, 2, 3, and -1.

The graph of the function $y = f(x)$ is helpful in visualizing the **real roots** (rational or irrational) of any equation $f(x) = 0$. The real roots of $f(x) = 0$ are the zeros of the function $f(x)$ and hence are the abscissas of the points where the graph of $y = f(x)$ crosses or touches the x-axis.

The y-intercept of the graph of $y = f(x)$ is the constant term in $f(x)$.

Whenever any number is tested for a root of the original equation by using the method of synthetic division, this number, along with the corresponding remainder, should be plotted. After a few points are plotted, they will serve as an aid in locating the roots. Additional points on the graph can readily be found by synthetic division by using any arbitrarily chosen values for x.

To illustrate, let us graph the function given in the equation $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$ of Example 2. To do this, let

$$y = x^4 + 2x^3 - 7x^2 - 8x + 12.$$

The y-intercept is 12, and from the results obtained in Example 2, we see that the x-intercepts are 1, 2, -2, and -3. This is sufficient information to draw an approximate graph. However, we shall in addition to these points find the value of $f(-1)$, thus:

$$\begin{array}{r|l} 1+2-7-8+12 & -1 \\ \quad -1-1+8+0 & \\ \hline 1+1-8+0+12 & \end{array} = f(-1).$$

The graph of $y = x^4 + 2x^3 - 7x^2 - 8x + 12$ is given in Fig. 28.

70. LOCATION OF REAL ROOTS

If the remainders, $f(a)$ and $f(b)$, are opposite in sign for the two real numbers a and b, then the equation $f(x) = 0$ has at least one real root or an odd number of real roots, between $x = a$ and $x = b$.
(A root of multiplicity m is counted as m roots.)

This is evident from the graph of $y = f(x)$ since the points P_1 and P_2, whose coördinates are $[a, f(a)]$ and $[b, f(b)]$, respectively, are on opposite sides of the x-axis. Hence any continuous curve connecting P_1 and P_2 must cross the x-axis at least once, or an odd number of times, between $x = a$ and $x = b$.*

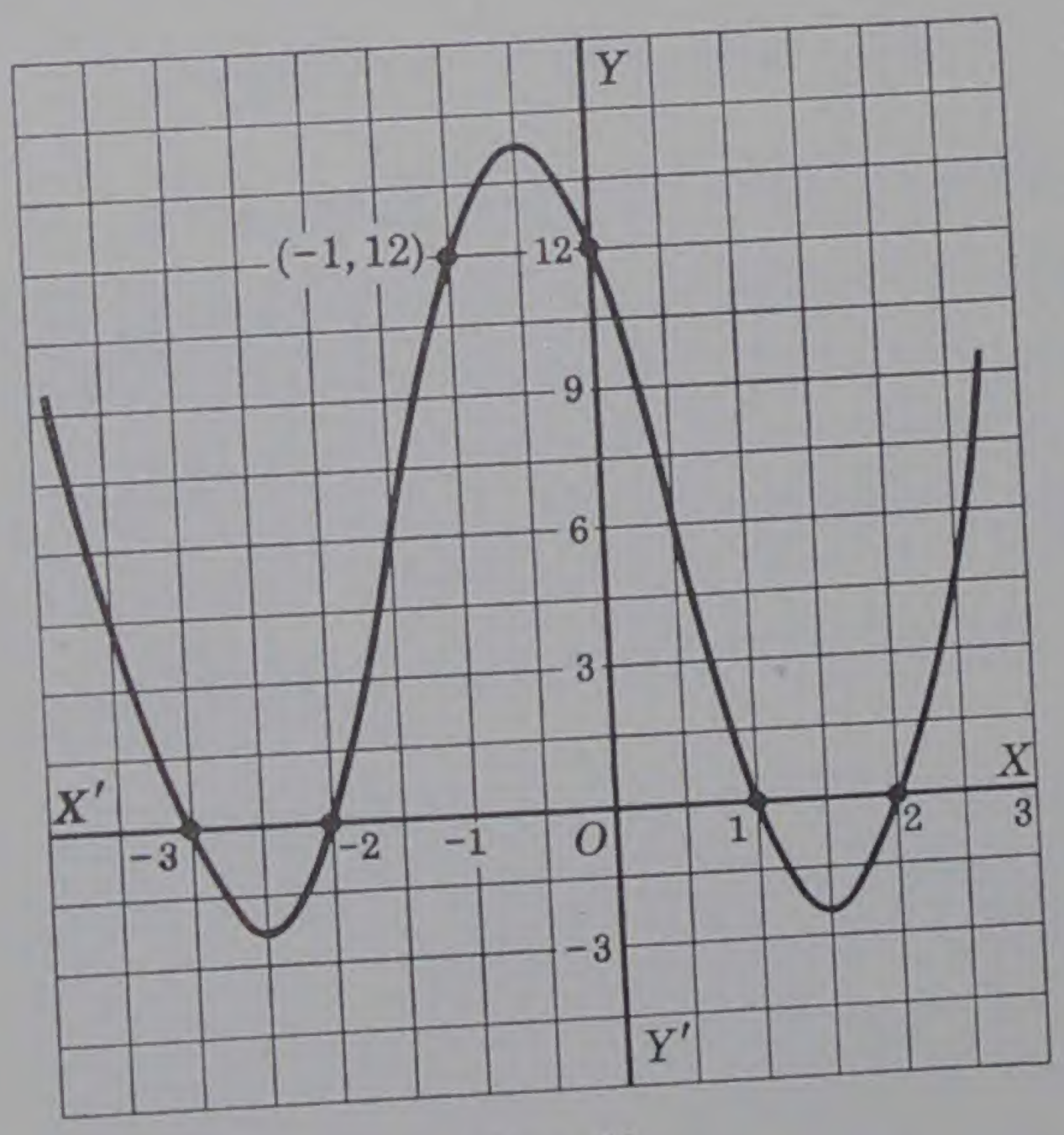


FIG. 28.

If $f(a)$ and $f(b)$ are alike in sign, the two points P_1 and P_2 are on the same side of the x-axis. Hence, any continuous curve connecting P_1 and P_2 either does not cross the x-axis, or must cross the x-axis an even number of times. Thus, *if the remainders, $f(a)$ and $f(b)$, are alike in sign for the two real numbers a and b, then the equation $f(x) = 0$ either has no real roots between $x = a$ and $x = b$, or it has an even number of real roots between $x = a$ and $x = b$.**

In an attempt to locate the roots of the equation $x^3 + 4x^2 + x - 6 = 0$, let us try the numbers -1, 2, and -4 by synthetic division; then

* This is visually obvious and we offer no proof.

$$\begin{array}{r|l} 1+4+1-6 & \underline{-1} \\ \quad -1-3+2 & \\ \hline 1+3-2-4 & = f(-1) \end{array} \qquad \begin{array}{r|l} 1+4+\ 1-\ 6 & \underline{2} \\ \quad +2+12+26 & \\ \hline 1+6+13+20 & = f(2) \end{array}$$

$$\begin{array}{r|l} 1+4+1-\ 6 & \underline{-4} \\ \quad -4+0-\ 4 & \\ \hline 1+0+1-10 & = f(-4) \end{array}$$

From this it follows that there is at least one real root between $x = -1$ and $x = 2$, because $f(-1) = -4$ and $f(2) = 20$ are opposite in sign. There is either no root or two real roots between $x = -1$ and $x = -4$, because $f(-1) = -4$ and $f(-4) = -10$ are alike in sign.

Let us now find the roots of the equation $x^3 + 4x^2 + x - 6 = 0$.

Testing 1, we have

$$\begin{array}{r|l} 1+4+1-6 & \underline{1} \\ \quad +1+5+6 & \\ \hline 1+5+6+0 & \end{array}$$

Therefore, $x = 1$ is a root.

From the depressed equation

$$x^2 + 5x + 6 = (x + 2)(x + 3) = 0,$$

we see that the other two roots are $x = -2$ and $x = -3$.

Therefore, the root between -1 and 2 is $x = 1$, and the two roots between -1 and -4 are $x = -2$ and $x = -3$.

71. UPPER AND LOWER LIMIT OF REAL ROOTS

There can be no real roots greater than a, a *being positive, if all the sums are positive in the synthetic division by* $x - a$, *nor can there be any roots smaller than* b, b *being negative, if the sums alternate in signs in the synthetic division by* $x - b$.

This is apparent because a number greater than a would merely increase each sum, including the remainder, and a number smaller than b would increase the numerical value of each of the sums.

In the illustration given in Article 70, all the signs in the sum are positive when $x^3 + 4x^2 + x - 6$ is divided by $x = 2$ by synthetic division. Hence, the equation $x^3 + 4x^2 + x - 6 = 0$ does not have any real roots greater than 2. The signs alternate in the sum when $x^3 + 4x^2 + x - 6$ is divided by $x = -4$. Hence, the equation $x^3 + 4x^2 + x - 6 = 0$ does not have any real roots less than -4 (the sign of 0 can be considered either positive or

negative). Thus, all the real roots of $x^3 + 4x^2 + x - 6 = 0$ must lie between -1 and $+2$.

72. A USEFUL THEOREM

If the constant term a_n of $f(x)$ has many factors, many trials may be required to find all the rational roots of $f(x) = 0$ or to show that $f(x) = 0$ does not have any rational roots. The following consideration is very useful to help exclude certain numbers as possible roots.

If r is a root of the equation $f(x) = 0$, then by the factor theorem

$$f(x) = (x - r)Q(x),$$

which is true for all values of x and hence must be true for, say, $x = m$. Therefore,

$$f(m) = (m - r)Q(m) \tag{18}$$

whenever r is a root of $f(x) = 0$.

THEOREM. *For a real number* r *to be a root of the equation* $f(x) = 0$, $(m - r)$ *must be a factor of* $f(m)$.

Example. Find all the rational roots of $f(x) = x^4 - 8x^3 + 6x + 40 = 0$.

Solution. All the rational roots of this equation must be integers and factors of 40. Hence, all the rational roots must be included in the following set of numbers, ± 1, ± 2, ± 4, ± 5, ± 8, ± 10, ± 20, and ± 40. Let us test to see whether or not 1 is a root: Note the use of 0 for the coefficient of x^2 since the x^2 term is missing in the given equation.

Testing 1, we have

$$\begin{array}{r|l} 1 - 8 + 0 + 6 + 40 & \underline{1} \\ \quad + 1 - 7 - 7 - 1 & \\ \hline 1 - 7 - 7 - 1 + 39 = f(1) & \end{array}$$

Since 1 is not a root, we may choose m to be 1 because we now have the value of $f(1)$. Let r assume the different values from the set of possible roots and test whether $m - r$ is a factor of $f(m) = f(1) = 39$. In this manner 5, 8, 10, 20, -1, -4, -5, -8, -10, -20, -40 are excluded as possible roots because $1 - r$ equals -4, -7, -9, -19, 2, 5, 6, 9, 11, 21, 41, respectively, and none of these numbers are factors of 39. This leaves only 2, 4, 40, and -2 to be tested for possible roots.

Testing 2, we have

$$\begin{array}{r|l} 1 - 8 + 0 + 6 + 40 & \underline{2} \\ \quad + 2 - 12 - 24 - 36 & \\ \hline 1 - 6 - 12 - 18 + 4 = f(2) & \end{array}$$

Since $f(2) = 4$, we can exclude 40 as a possible root. Why?

Testing 4, we have

$$\begin{array}{r|l} 1 - 8 + 0 + 6 + 40 & \underline{4} \\ \quad + 4 - 16 - 64 - 232 & \\ \hline 1 - 4 - 16 - 58 - 192 = f(4) & \end{array}$$

Testing -2, we have

$$\begin{array}{r|l} 1 - 8 + 0 + 6 + 40 & \underline{-2} \\ \quad - 2 + 20 - 40 + 68 & \\ \hline 1 - 10 + 20 - 34 + 108 = f(-2) & \end{array}$$

Therefore, $f(x) = x^4 - 8x^3 + 6x + 40 = 0$ does not have any rational roots. However, this equation does have an irrational root between 2 and 4 because $f(2)$ and $f(4)$ are opposite in sign.

Testing 8, we have

$$\begin{array}{r|l} 1 - 8 + 0 + 6 + 40 & \underline{8} \\ \quad + 8 + 0 + 0 + 48 & \\ \hline 1 + 0 + 0 + 6 + 88 = f(8) & \end{array}$$

Since $f(4)$ and $f(8)$ are opposite in sign, there is another irrational root between 4 and 8. Irrational roots will be studied in Article 77.

Many of the numbers could have been excluded as possible roots of this equation by using the upper and lower limits of roots. Since the signs of the sums alternate in the synthetic division by $x = -2$, there cannot be a root less than -2. Since the signs of the sums are all positive in the synthetic division by $x = 8$, there cannot be any roots greater than 8. (The student should try numbers smaller than -2 and greater than 8 and observe that the numerical values of the respective sums do increase.)

73. DESCARTES'S RULE OF SIGNS

Another helpful consideration when solving algebraic equations is the determination of the maximum number of positive roots and the maximum number of negative roots any algebraic equation $f(x) = 0$ can have. The maximum number of positive roots any algebraic equation $f(x) = 0$ can have, can be determined from the number of variations of sign of $f(x)$. (We shall give only a partial proof of Descartes's rule of signs.)

If the terms of $f(x)$ are arranged in descending powers of x, then wherever the coefficients of two adjacent terms have different signs, we say $f(x)$ has a **variation in sign.**

Let us multiply any polynomial $f(x)$ by $x - r$ (r positive), where $f(x)$ is arranged in descending powers of x, and let us suppose that the signs of $f(x)$ and $x - r$ are as indicated in the first two

lines below. The signs of the products are shown by the subsequent lines.

$$\begin{array}{rl} f(x): & + + + - - - + + + - + - \\ x - r: & + - \\ \hline f(x) \times x: & + + + - - - + + + - + - \\ f(x) \times -r: & \quad - - - + + + - - - + - + \\ \hline (x - r)f(x) = F(x): & + \quad - \quad + \quad - + - + \end{array} \tag{19}$$

Thus we obtain the known signs of the product $(x - r)f(x) = F(x)$ for this special case.

From this special product we can observe the following facts concerning the general product $F(x) = (x - r)f(x)$ for any polynomial $f(x)$:

1. The sign of the coefficient of the highest power of x in the product $F(x)$ is the same as the sign of the coefficient of the highest power of x in $f(x)$.

2. If the $(k - 1)$st and the kth terms in $f(x)$ are alike in sign, then the sign of the kth term in the product $F(x)$ is in doubt.

3. If the $(k - 1)$st and the kth terms in $f(x)$ are opposite in sign, then the sign of the kth term in $f(x)$ and the sign of the kth term in $F(x)$ are alike in sign.

4. The last term in $F(x)$ and the last term in $f(x)$ are opposite in sign.

In practice we count the number of variations of sign in $f(x)$ without supplying any signs for the missing terms. However, in obtaining the results in Equation (19) we assumed that each missing term had a sign. For this discussion, if all the coefficients between two negative terms are zero we assume that the missing coefficients are -0, and the coefficients of the remaining missing terms are $+0$.

Since the equation $F(x) = (x - r)f(x) = 0$ has one more positive root than $f(x) = 0$, and we have shown that the function $F(x)$ has at least one more variation in sign than $f(x)$, therefore, for *every positive root of an algebraic equation* $f(x) = 0$ *with real coefficients there must be at least one variation in sign in* $f(x)$. However, the converse need not be true. The following theorem was first proved by Descartes in 1637, and is known as Descartes's rule of signs.

DESCARTES'S RULE OF SIGNS. *The algebraic equation* $f(x) = 0$ *with real coefficients cannot have more positive roots than* $f(x)$ *has*

variations of sign, and the number of positive roots of $f(x) = 0$ *is either equal to the number of variations of sign of* $f(x)$ *or is less than this number by an even integer.*

By substituting $-x$ for x in $y = f(x)$, the graph of the resulting function $y = f(-x)$ will be the graph of $y = f(x)$ reflected in the y-axis, or, what is the same thing, will be the graph of $y = f(x)$ rotated through 180° about the y-axis. Hence, the positive roots of $f(-x) = 0$ are the negative roots of $f(x) = 0$, and the negative roots of $f(-x) = 0$ are the positive roots of $f(x) = 0$. Applying Descartes's rule of signs to $f(-x)$ for the negative roots of $f(x)$, we have that:

The equation $f(x) = 0$ *cannot have more negative roots than* $f(-x)$ *has variations of sign, and the number of negative roots of* $f(x) = 0$ *is either equal to the number of variations of sign of* $f(-x)$ *or is less than this number by an even integer.* [The student should not confuse $f(-x)$ with $-f(x)$.]

74. RATIONAL FRACTIONAL ROOTS

When testing to see whether a fraction is a root by using the method of synthetic division, the work need not be carried beyond the point where fractions are introduced. To illustrate:

Example. Find the rational roots of $12x^3 - 4x^2 - 17x - 6 = 0$.

Solution. All the rational roots must be included in the following set of numbers, $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{1}{4}, \pm \frac{3}{4}, \pm \frac{1}{6}, \pm \frac{1}{12}$.

Testing to see whether $\frac{1}{2}$ is a root or not, we have

$$\begin{array}{rrrrr} 12 & -4 & -17 & -6 & \lfloor \frac{1}{2} \\ & +6 & +1 & -8 & \\ \hline 12 & +2 & -16 & -14 & \end{array}$$

hence $\frac{1}{2}$ is not a root.

Testing to see whether $\frac{2}{3}$ is a root, we have

$$\begin{array}{rrrrr} 12 & -4 & -17 & -6 & \lfloor \frac{2}{3} \\ & +8 & & & \\ \hline 12 & +4 & & & \end{array}$$

The work need not be carried any farther, because if fractions are introduced for any sum, then the remaining sums will contain fractions and the remainder cannot be zero.

Testing $\frac{3}{2}$

$$\begin{array}{rrrrr} 12 & -4 & -17 & -6 & \lfloor \frac{3}{2} \\ & +18 & +21 & +6 & \\ \hline 12 & +14 & +4 & 0 & \end{array}$$

hence $\frac{3}{2}$ is a root, and we may now write

$$12x^3 - 4x^2 - 17x - 6 = (x - \tfrac{3}{2})(12x^2 + 14x + 4)$$

or

$$= (2x - 3)(6x^2 + 7x + 2);$$

factoring

$$= (2x - 3)(3x + 2)(2x + 1).$$

Therefore, the roots of the equation $12x^3 - 4x^2 - 17x - 6 = 0$ are $\frac{3}{2}$, $-\frac{2}{3}$, and $-\frac{1}{2}$.

75. IMAGINARY ROOTS

If

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \qquad (20)$$

has an imaginary root, say $a + bi$, then $f(a + bi)$ must be zero, and we have

$$f(a + bi) = a_0(a + bi)^n + a_1(a + bi)^{n-1} + \cdots + a_{n-1}(a + bi) + a_n = 0, \qquad (21)$$

where we shall assume all the coefficients a_i, $(i = 0, 1, 2 \cdots n)$ to be real numbers.

If the indicated powers of $a + bi$ are taken and the higher powers of i are reduced (Article 54), then Equation (21) will be composed of real terms free of i and imaginary terms containing i. Let P denote the sum of all the terms free of i and Q the sum of the coefficients of i; then we may write

$$f(a + bi) = P + Qi = 0. \qquad (22)$$

Since P and Q are real, it follows from Article 54 that for $P + Qi$ to equal zero it is necessary that

$$P = 0 \quad \text{and} \quad Q = 0. \qquad (23)$$

Let us next consider the value of $f(x)$ when $a - bi$ has been substituted for x; then

$$f(a - bi) = a_0(a - bi)^n + a_1(a - bi)^{n-1} + \cdots + a_{n-1}(a - bi) + a_n. \qquad (24)$$

Now Equation (24) is nothing more than Equation (21) with $-i$ substituted for $+i$. Since all even powers of i are real, it follows that the real parts of both Equations (21) and (24) are equal if the coefficients of $f(x)$ are real. Since all odd powers of i are imaginary, and the odd powers of $-i$ are equal but opposite in

sign to the same odd powers of $+i$, it follows that the sum of the coefficients of i in Equation (24) is the negative of the sum of the coefficients of i in Equation (21). Thus,

$$f(a - bi) = P - Qi.$$

But since $P = Q = 0$,

$$f(a - bi) = P - Qi = 0. \tag{25}$$

Thus we can state the following theorem:

IMAGINARY ROOTS. *If the imaginary number* $a + bi$ *is a root of the algebraic equation* $f(x) = 0$ *with real coefficients, then the conjugate imaginary number* $a - bi$ *is also a root.* Therefore, *the total number of imaginary roots of an algebraic equation with real coefficients must be an even number.*

We shall make no attempt to find the imaginary roots of any equation unless the quadratic factors of $f(x)$ can readily be found. If the depressed equation is a quadratic, then this quadratic equation can always be solved by the quadratic formula, whether the two roots are real or imaginary, rational or irrational.

Summary

We shall now illustrate how the preceding theorems can be used to solve equations.

Example. Solve $f(x) = 6x^4 + x^3 - 5x^2 - 11x - 6 = 0$.

Solution. Since $f(x)$ has only one variation in sign, the given equation must have one positive root. From the equation

$$f(-x) = 6x^4 - x^3 - 5x^2 + 11x - 6 = 0,$$

we see that $f(-x)$ has three variations of sign. Hence, the given equation either has three negative roots or it has only one negative root.

Therefore, the given equation has either one positive root and three negative roots, or it has one positive root, one negative root, and two imaginary roots.

The rational roots of this equation must be included in the following set of numbers

$$\begin{gathered}
\pm 1, \quad \pm 2, \quad \pm 3, \quad \pm 6, \\
\pm \frac{1}{2}, \quad \pm \frac{2}{2}, \quad \pm \frac{3}{2}, \quad \pm \frac{6}{2}, \\
\pm \frac{1}{3}, \quad \pm \frac{2}{3}, \quad \pm \frac{3}{3}, \quad \pm \frac{6}{3}, \\
\pm \frac{1}{6}, \quad \pm \frac{2}{6}, \quad \pm \frac{3}{6}, \quad \pm \frac{6}{6}.
\end{gathered}$$

Testing 1, we have

$$\begin{array}{r|l} 6+1-5-11-6 & 1 \\ \quad +6+7+2-9 & \\ \hline 6+7+2-9-15 & \end{array}$$

One is not a root.

Testing 2, we have

$$\begin{array}{r|l} 6+1-5-11-6 & 2 \\ \quad +12+26+42+62 & \\ \hline 6+13+21+31+56 & \end{array}$$

The positive root must be less than 2 since all the signs of the sums are positive. Furthermore, the positive root must lie between 1 and 2 because $f(1)$ and $f(2)$ are opposite in sign. The only number between 1 and 2 in the foregoing set of numbers, which contains all the rational roots, is $\frac{3}{2}$.

Testing $\frac{3}{2}$, we have

$$\begin{array}{r|l} 6+1-5-11-6 & \frac{3}{2} \\ \quad +9+15+15+6 & \\ \hline 6+10+10+4+0 & \end{array}$$

Therefore $\frac{3}{2}$ is a root, and we may write

$$6x^4+x^3-5x^2-11x-6=(x-\tfrac{3}{2})(6x^3+10x^2+10x+4)$$

or

$$=(2x-3)(3x^3+5x^2+5x+2).$$

We shall now solve the depressed equation $3x^3+5x^2+5x+2=0$. But first let us test -1 in the original equation.

Testing -1, we have

$$\begin{array}{r|l} 6+1-5-11-6 & -1 \\ \quad -6+5+0+11 & \\ \hline 6-5+0-11+5 & \end{array}$$

There are no negative roots less than -1, since the signs of the sums alternate. Since $f(0)=-6$ and $f(-1)=5$ are opposite in sign, there must be either *one* root or *three* roots between 0 and -1.

Since the leading coefficient in the depressed equation is 3, we need try only some thirds for roots.

Testing $-\frac{1}{3}$, we have

$$\begin{array}{r|l} 3+5+5+2 & -\frac{1}{3} \\ \quad -1 & \\ \hline 3+4 & \end{array}$$

This shows that $-\frac{1}{3}$ is not a root.

Testing $-\frac{2}{3}$, we have

$$\begin{array}{r|l} 3+5+5+2 & -\frac{2}{3} \\ -2-2-2 & \\ \hline 3+3+3+0 & \end{array}$$

Therefore $-\frac{2}{3}$ is a root, and we may now write the given equation as:

$$\begin{aligned} f(x) &= 6x^4 + x^3 - 5x^2 - 11x - 6 \\ &= (2x-3)(x+\tfrac{2}{3})(3x^2+3x+3) = 0 \\ f(x) &= (2x-3)(3x+2)(x^2+x+1) = 0, \end{aligned}$$

and the four roots are

$$x = \frac{3}{2}, \quad -\frac{2}{3}, \quad \frac{-1+\sqrt{3}i}{2}, \quad \frac{-1-\sqrt{3}i}{2},$$

since the roots of $x^2+x+1=0$ are $\dfrac{-1\pm\sqrt{3}i}{2}$.

The graph of $y = 6x^4 + x^3 - 5x^2 - 11x - 6$ is given in Fig. 29.

Had we divided the depressed equation by -1, as follows,

$$\begin{array}{r|l} 3+5+5+2 & -1 \\ -3-2-3 & \\ \hline 3+2+3-1 & \end{array}$$

then the remainder -1 would not be the value of $f(-1)$ of the original function. However, since

$$f(x) = (2x-3)(3x^3+5x^2+5x+2),$$

then

$$f(-1) = (-2-3)(-1) = 5.$$

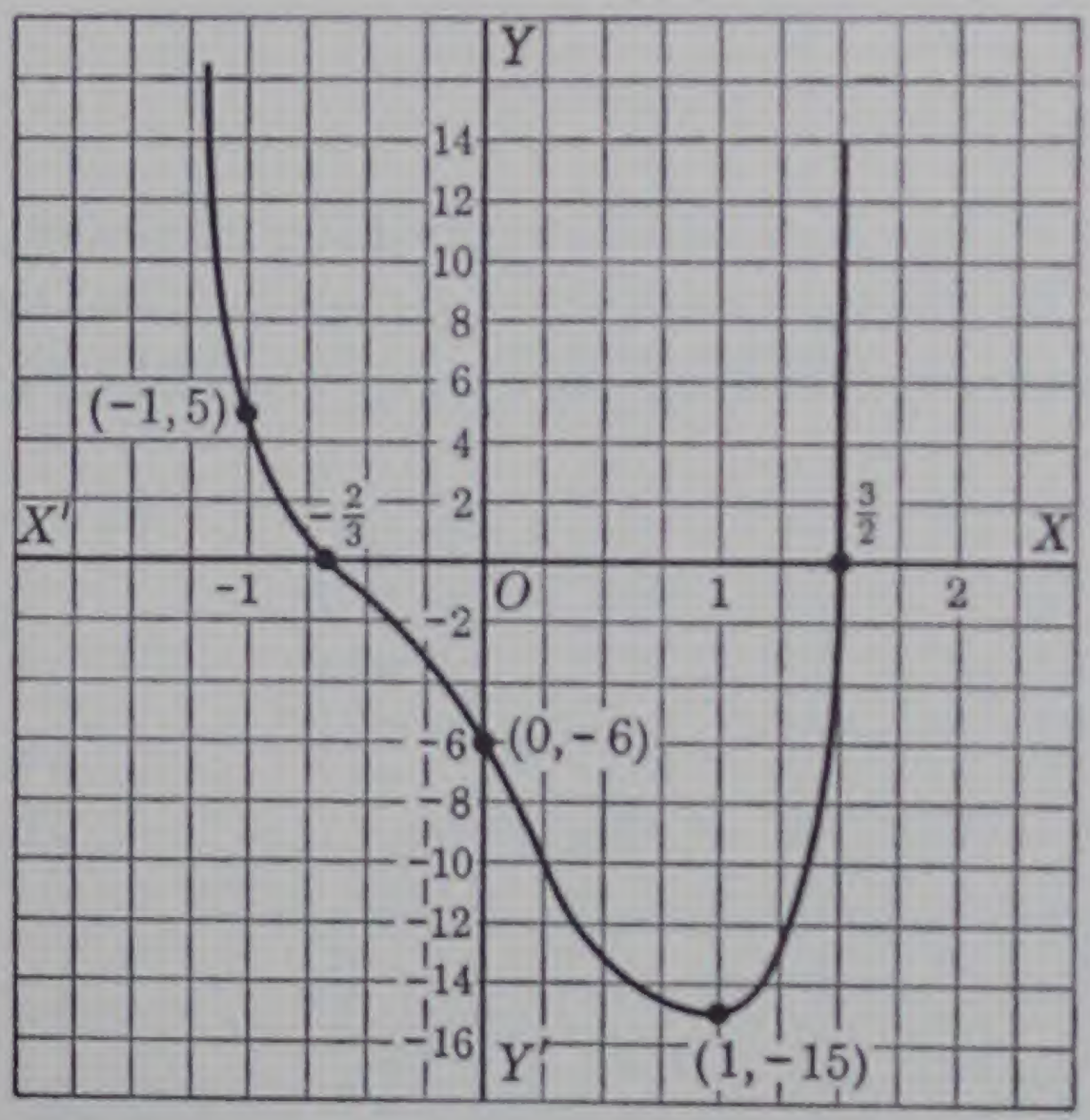


FIG. 29.

(5 was the remainder when we divided by -1 above.) After we have found a root r of the given equation $f(x)=0$, then the value of $f(m)$ equals $(m-r)$ times the remainder obtained by dividing the depressed equation by $x-m$. This is evident from the following since

$$f(x) = (x-r)Q(x); \qquad \text{(by Equation 10)}$$

then

$$f(m) = (m-r)Q(m).$$

QUESTIONS

1. What is synthetic division?
2. What is synthetic substitution?
3. How can a real root be located?
4. What is an upper bound for real roots?
5. What is a lower bound for real roots?
6. What is Descartes's rule of signs?
7. What can be said about the number of imaginary roots of an algebraic equation?

EXERCISES

By Descartes's rule of signs, determine the largest possible number of positive roots and negative roots each of the equations in Exercises 1 and 2 can have; hence, at least how many imaginary roots must each of the equations have?

1. (a) $5x^6 - 6x^2 + 2 = 0$; (b) $x^8 + 5x^3 + 10 = 0$;
   (c) $6x^7 - 2x^5 - x^2 - 6 = 0$; (d) $x^3 - 1 = 0$.
2. (a) $3x^6 - 4x^2 - 2x + 5 = 0$; (b) $2x^8 + 6x^3 + 2x - 7 = 0$;
   (c) $x^6 + 64 = 0$; (d) $x^n - a^n = 0$.

Solve each of the equations in Exercises 3 through 48 for all the roots. Find all the rational roots first. Give the rational factors of each function and also draw the graph of each function.

3. $x^3 - 5x^2 - 4x + 20 = 0$.
4. $x^3 - 9x^2 + 23x - 15 = 0$.
5. $x^3 - 5x^2 - 2x + 24 = 0$.
6. $x^3 + 12x^2 + 44x + 48 = 0$.
7. $x^3 + 6x^2 + 5x - 12 = 0$.
8. $x^3 - 1 = 0$.
9. $x^3 - 7x - 6 = 0$.
10. $x^3 - 2x^2 - 5x + 6 = 0$.
11. $x^4 + 7x^3 - 36x = 0$.
12. $x^3 - 6x - 4 = 0$.
13. $x^4 + 2x^3 - 11x^2 - 12x + 36 = 0$.
14. $x^3 - 2x^2 - 4x + 8 = 0$.
15. $x^3 - x^2 - x - 2 = 0$.
16. $x^4 - 2x^3 - 17x^2 + 18x + 72 = 0$.
17. $x^3 + x^2 - 2x + 12 = 0$.
18. $x^4 - 5x^2 - 8x - 12 = 0$.
19. $x^3 + 3x^2 - 2x - 6 = 0$.
20. $x^3 - 35x - 6 = 0$.
21. $x^3 - 3x^2 - 9x + 2 = 0$.
22. $x^4 + 2x^3 - 19x^2 - 68x - 60 = 0$.
23. $x^3 - 2x^2 - 4x + 3 = 0$.
24. $x^3 + x^2 - 20x = 0$.
25. $x^3 + 12x^2 + 44x + 48 = 0$.

26. $x^4 + 3x^3 + x^2 - 15x - 14 = 0.$
27. $6x^3 - 17x^2 - 5x + 6 = 0.$
28. $4x^4 - 8x^3 + x^2 - 3x + 9 = 0.$
29. $2x^3 + 3x^2 - 11x - 6 = 0.$
30. $16x^4 - 24x^2 + 16x - 3 = 0.$
31. $6x^3 - 5x^2 - 22x + 24 = 0.$
32. $3x^3 + 8x^2 - x - 10 = 0.$
33. $6x^4 - 13x^3 - x^2 - 2x - 20 = 0.$
34. $6x^3 - 5x^2 - 37x - 6 = 0.$
35. $4x^3 - 33x - 20 = 0.$
36. $6x^3 - 35x^2 + 19x + 30 = 0.$
37. $9x^4 - 22x^2 + 24x - 8 = 0.$
38. $12x^3 - 4x^2 - 53x - 30 = 0.$
39. $4x^4 - 8x^3 - 5x^2 + 16x - 6 = 0.$
40. $36x^4 - 12x^3 - 101x^2 + 17x + 30 = 0.$
41. $3x^3 - 4x^2 - x + 2 = 0.$
42. $3x^4 - 12x^3 + 14x^2 - 8x + 8 = 0.$
43. $x^3 + 7x^2 - 14x - 120 = 0.$
44. $x^4 + x^3 + 13x^2 + 25x - 300 = 0.$
45. $x^4 - 37x^2 - 24x + 180 = 0.$
46. $x^4 + x^3 + 12x - 144 = 0.$
47. $x^4 - 190x^2 - 864x + 360 = 0.$
48. $x^4 - 41x^2 + 400 = 0.$

76. TRANSFORMATION OF EQUATIONS

If all the coefficients of the equation

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n = 0 \qquad (26)$$

are integers, and a_0 and a_n have many factors, then it may take many trials to find all the rational roots because there are many possible roots in the set of numbers containing all the rational roots.

Sometimes the work for finding rational roots is simplified if the given equation is transformed into a new equation in which the coefficient of the highest power of the new unknown is 1 and the remaining coefficients are integers, for then the rational roots of the transformed equation must be integers. This transformation is obtained by substituting $\frac{y}{m}$ for x in the given equation. Thus, substituting $\frac{y}{m}$ for x in Equation (26), we have

$$f\left(\frac{y}{m}\right) = a_0\left(\frac{y}{m}\right)^n + a_1\left(\frac{y}{m}\right)^{n-1} + a_2\left(\frac{y}{m}\right)^{n-2} + \cdots + a_{n-2}\left(\frac{y}{m}\right)^2 + a_{n-1}\left(\frac{y}{m}\right) + a_n = 0. \qquad (27)$$

Clearing this equation of fractions by multiplying each term by m^n, we have

$$a_0y^n + a_1y^{n-1}m + a_2y^{n-2}m^2 + \cdots + a_{n-2}y^2m^{n-2} + a_{n-1}ym^{n-1} + a_nm^n = 0. \qquad (28)$$

Since $\frac{y}{m} = x$ implies $y = mx$, it follows that *the roots of the transformed equation are* m *times the roots of the original equation* $f(x) = 0$.

It is always possible to choose m so that each term of Equation (28) can be divided by a_0 without introducing any fractions. Thus it is always possible to transform any algebraic equation into a new equation which is such that all the coefficients in the transformed equation are integers and the coefficient of the highest power of the unknown is 1, and hence all the rational roots of the transformed equation are integers.

We observe from Equation (28) *that the equation whose roots are* m *times the roots of a given equation can be formed from the original equation* $f(x) = 0$ *by multiplying the first term by* m^0, *the second term by* m, *the third term by* m^2, *and so on (supplying zero for the missing terms), and that each term of the transformed equation is of degree* n *in* m *and* y *(the sum of the exponents of* m *and* y *in each term is* n*).*

To illustrate the application of this principle, let us solve the example in Article 74 for rational roots by first transforming the equation

$$12x^3 - 4x^2 - 17x - 6 = 0$$

into

$$12y^3 - 4y^2m - 17ym^2 - 6m^3 = 0.$$

Let $m = 6$; then by substitution

$$12y^3 - 4 \cdot 6y^2 - 17 \cdot 6^2y - 6 \cdot 6^3 = 0;$$

dividing by 12, we have

$$F(y) = y^3 - 2y^2 - 51y - 108 = 0.$$

In this transformed equation the coefficient of the highest power of the unknown is 1, and hence the rational roots of the transformed equation are included in the set of numbers ± 1, ± 2, ± 3, ± 4, ± 6, ± 9, ± 12, ± 18, ± 27, ± 36, ± 54, ± 108. By Descartes's rule of signs, this equation must have one positive root.

Testing 1, we have

$$\begin{array}{r} 1 - 2 - 51 - 108 \quad \underline{|1} \\ +\,1 - \;\; 1 - \;\; 52 \\ \hline 1 - 1 - 52 - 160 = F(1) \end{array}$$

Since $F(1) = -160$, then by Article 72, 4, 12, 18, 27, 36, 54, 108, -2, -6, -12, -18, -27, -36, -54, -108 cannot be roots of the transformed equation because $(1 - r)$ equals -3, -11, -17, -26, -35, -53, -107, 3, 7, 13, 19, 28, 37, 55, 109, and none of these numbers are factors of -160. This leaves only 2, 3, 6, 9, -1, -3, -4, -9 to be tested for possible roots.

Testing 2, we have

$$\begin{array}{r} 1 - 2 - 51 - 108 \quad \underline{|2} \\ +\,2 + \;\; 0 - 102 \\ \hline 1 + 0 - 51 - 210 = F(2) \end{array}$$

Since $F(2) = -210$, then 6 and -9 cannot be roots because $(2 - r)$ equals -4 and 11 and these numbers are not factors of -210.

Testing 3 and 9, we have

$$\begin{array}{r} 1 - 2 - 51 - 108 \quad \underline{|3} \\ +\,3 + \;\; 3 - 144 \\ \hline 1 + 1 - 48 - 252 \end{array} \qquad \begin{array}{r} 1 - 2 - 51 - 108 \quad \underline{|9} \\ +\,9 + 63 + 108 \\ \hline 1 + 7 + 12 + \;\; 0 \end{array}$$

Hence, 9 is a root, and

$$\begin{aligned} y^3 - 2y^2 - 51y - 108 &= (y - 9)(y^2 + 7y + 12) \\ &= (y - 9)(y + 4)(y + 3) \\ &= 0. \end{aligned}$$

Thus the roots of the transformed equation are $y = 9, -4$, and -3.

Since we let $m = 6$, the roots of the transformed equation are 6 times the roots of the original equation. Thus the roots of the original equation are $x = \frac{y}{m} = \frac{9}{6}, -\frac{4}{6}$, and $-\frac{3}{6}$, or $x = \frac{3}{2}, -\frac{2}{3}$, and $-\frac{1}{2}$.

Transformations

(1). *The equation whose roots are* m *times the roots of the given equation* $f(x) = 0$ *is* $f\left(\frac{y}{m}\right) = 0$. *Proof:* Since the unknown y in the new equation is to have the value mx, then $x = \frac{y}{m}$. Therefore the new equation is obtained by substituting $\frac{y}{m}$ for x in the given equation.

(2). *The equation whose roots are* h *units more than the roots of the given equation* $f(x) = 0$ *is* $f(y - h) = 0$. *Proof:* Since the unknown y in the new equation is to have the value $x + h$, then $x = y - h$. Therefore the new equation is obtained by substituting $y - h$ for x in the given equation.

(3). *The equation whose roots are* h *units less than the roots of the given equation* $f(x) = 0$ *is* $f(y + h) = 0$. *Proof:* Since the unknown y in the new equation is to have the value $x - h$, then $x = y + h$. Therefore, the new equation is obtained by substituting $y + h$ for x in the given equation.

We have previously shown in Equation (28) how the equation whose roots are m times the roots of a given equation can be obtained.

We shall now show how the equation whose roots are h units less than the roots of a given equation can be obtained.

Let the given equation be

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0;$$

then the transformed equation is

$$f(y + h) = a_0(y + h)^n + a_1(y + h)^{n-1} + \cdots + a_{n-1}(y + h) + a_n = 0. \qquad (29)$$

The process of expanding and collecting the terms in this equation may be long and laborious. However, from the following consideration, this new equation may be derived in the following manner.

Let us suppose the transformed Equation (29), if simplified, reduces to

$$f(y + h) = a_0y^n + R_ny^{n-1} + R_{n-1}y^{n-2} + \cdots + R_2y + R_1 = 0. \qquad (30)$$

Now if we can determine these coefficients, the R's, then we will have the transformed equation.

Since $y = x - h$, we have, upon substituting $x - h$ for y in Equation (30),

$$f(y + h) = f(x) = a_0(x - h)^n + R_n(x - h)^{n-1} + \cdots + R_2(x - h) + R_1 = 0. \quad (31)$$

From this equation it is evident that R_1 is the remainder obtained by dividing $f(x)$ by $x - h$, R_2 is the remainder obtained by dividing the quotient by $x - h$, and so forth. That is, the R's are the successive remainders obtained by dividing $f(x)$ and the successive quotients by $x - h$. These remainders can be found by using synthetic division repeatedly. The method for obtaining these remainders is best illustrated by the following example.

Example 1. Find the equation whose roots are 2 units less than the roots of the equation $f(x) = x^3 + 4x^2 + x - 6 = 0$.

Solution. We shall obtain the transformed equation by each of the two methods, substitution and synthetic division.

By the substitution method we substitute $y + 2$ for x in $f(x) = 0$. Thus we have

$$(y + 2)^3 + 4(y + 2)^2 + (y + 2) - 6 = 0;$$

expanding gives

$$(y^3 + 6y^2 + 12y + 8) + 4(y^2 + 4y + 4) + y + 2 - 6 = 0;$$

hence

$$y^3 + 6y^2 + 12y + 8 + 4y^2 + 16y + 16 + y + 2 - 6 = 0;$$

collecting terms gives

$$y^3 + 10y^2 + 29y + 20 = 0.$$

By the method of synthetic division we divide $f(x)$ by $x - 2$. Then:

$$\begin{array}{rrrrl} 1 + & 4 + & 1 - & 6 & \underline{|2} \\ & + & 2 + & 12 + & 26 \\ \hline 1 + & 6 + & 13 + & \mathbf{20} & = R_1 \\ & + & 2 + & 16 & \\ \hline 1 + & 8 + & \mathbf{29} & & = R_2 \\ & + & 2 & & \\ \hline 1 + & \mathbf{10} & & & = R_3 \end{array}$$

$$Q_1(x) = x^2 + 6x + 13,$$

$$Q_2(x) = x + 8,$$

$$a_0 = Q_3(x) = 1.$$

Therefore the equation whose roots are 2 units less than the roots of $x^3 + 4x^2 + x - 6 = 0$ is $y^3 + 10y^2 + 29y + 20 = 0$. As an exercise, the student should solve the two equations $x^3 + 4x^2 + x - 6 = 0$ and $y^3 + 10y^2 + 29y + 20 = 0$ and see that the three roots of the latter

equation are respectively 2 units less than the three roots of the first equation.

Example 2. Find the equation whose roots are 4 units more than the roots of the equation $f(x) = x^3 + 4x^2 + x - 6 = 0$.

Solution. In this case we must divide $f(x)$ by $x + 4$. Hence

$$\begin{array}{rrrrl} 1 & +4 & +1 & -6 & \lfloor\underline{-4} \\ & -4 & +0 & -4 & \\ \hline 1 & +0 & +1 & \mathbf{-10} & = R_1 \\ & -4 & +16 & & \\ \hline 1 & -4 & \mathbf{+17} & & = R_2 \\ & -4 & & & \\ \hline 1 & \mathbf{-8} & & & = R_3 \end{array}$$

Therefore the equation whose roots are 4 units more than the roots of $x^3 + 4x^2 + x - 6 = 0$ is $y^3 - 8y^2 + 17y - 10 = 0$. The student should solve these two equations and see that the three roots of the latter equation are respectively 4 units more than the three roots of the first equation.

QUESTIONS

1. What is the equation whose roots are m times the roots of the equation $f(x) = 0$?

2. What is the equation whose roots are h units more than the roots of the equation $f(x) = 0$?

3. What is the equation whose roots are h units less than the roots of the equation $f(x) = 0$?

EXERCISES

1. Give the equation whose roots are six times the roots of $6x^3 - 17x^2 - 5x + 6 = 0$.

2. Give the equation whose roots are twice the roots of $x^3 - 3x^2 + 2x - 1 = 0$.

3. Give the equation whose roots are twice the roots of $2x^3 + 3x^2 - 11x - 6 = 0$.

4. Give the equation whose roots are three times the roots of $2x^3 - x + 5 = 0$.

5. Give the equation whose roots are twice the roots of $3x^4 - 5x + 1 = 0$.

6. Give the equation whose roots are three times the roots of $9x^4 - 2x^2 - 4x - 2 = 0$.

7. Give the equation whose roots are $\frac{1}{3}$ the roots of $2x^3 - x^2 + 3x - 5 = 0$.

8. Give the equation whose roots are $\frac{1}{3}$ the roots of $2x^4 - 5x^2 + 9 = 0$,

9. Give the equation whose roots are the negative of the roots of $6x^5 - 4x^3 - 6x^2 + x - 5 = 0$.

10. Give the equation whose roots are the negatives of the roots of $2x^7 - 5x^4 + 3x^2 + x - 1 = 0$.

11. Give the equation whose roots are the reciprocals of the roots of $2x^4 - 3x^2 + 2x + 4 = 0$.

12. Give the equation whose roots are the reciprocals of the roots of $3x^7 - 5x^5 + 2x^4 - 3x + 7 = 0$.

13. Give the equation whose roots are 3 less than the roots of $x^3 - 5x^2 + 3x + 6 = 0$.

14. Give the equation whose roots are 2 less than the roots of $2x^3 - 5x + 2 = 0$.

15. Give the equation whose roots are 1 more than the roots of $x^3 + 6x^2 + 3x + 5 = 0$.

16. Give the equation whose roots are 2 more than the roots of $x^3 + x^2 + x + 1 = 0$.

17. Give the equation whose roots are 0.2 less than the roots of $2x^3 + 9x^2 + 12x + 5 = 0$.

18. Give the equation whose roots are -4 less than the roots of $x^3 + 3x^2 - 9x - 5 = 0$.

19. Give the equation whose roots are 1 more than the roots of $x^3 - 2x^2 + 3x - 4 = 0$.

20. Give the equation whose roots are $\frac{1}{2}$ the roots of $x^3 - 3x^2 + 2x - 1 = 0$.

Find all the roots of each of the following equations.

21. $x^4 - 37x^2 - 24x + 180 = 0$.

22. $12x^4 + 8x^3 - 15x^2 - 28x - 12 = 0$.

23. $x^4 + x^3 + 13x^2 + 25x - 300 = 0$.

24. $6x^4 - x^3 - 480x^2 - 1120x + 200 = 0$.

25. $x^4 - 190x^2 - 864x + 360 = 0$.

26. $4x^3 - x^2 - 36x + 9 = 0$.

27. $3x^3 + 7x^2 - 36x - 84 = 0$.

28. $12x^3 - 73x^2 + 36 = 0$.

29. $9x^3 - 9x^2 - 190x + 400 = 0$.

30. $2x^4 - x^3 - 10x^2 + 2x + 12 = 0$.

77. IRRATIONAL ROOTS

There are several methods devised for finding the irrational roots of a numerical equation to any desired number of decimal places.

Horner's Method

The method most commonly used, though perhaps not the shortest, was published in 1819 by W. G. Horner (1786–1837), an

English mathematician, and is known as Horner's method for approximating irrational roots. The procedure for approximating the irrational roots is as follows:

STEP 1. Locate a root between two consecutive integers, say a_1 and $a_1 \pm 1$. Thus $x_1 = a_1$ is the first approximation of the given root. In actual practice a_1 can be either larger or smaller than the root. However, in the following discussion we shall suppose a_1 to be positive and smaller than the root.

STEP 2. By means of transformation (3) given in Article 76, find a new equation whose roots are a_1 units less than the roots of the given equation. Graphically this means to change the origin to the point $(a_1, 0)$.

STEP 3. The first transformed equation will then have a root between 0 and 1. Continue by locating this root between two consecutive tenths, say between $0.a_2$ and $0.a_2 + 0.1$. The approximate root of the first transformed equation is then $x_2 = 0.a_2$.

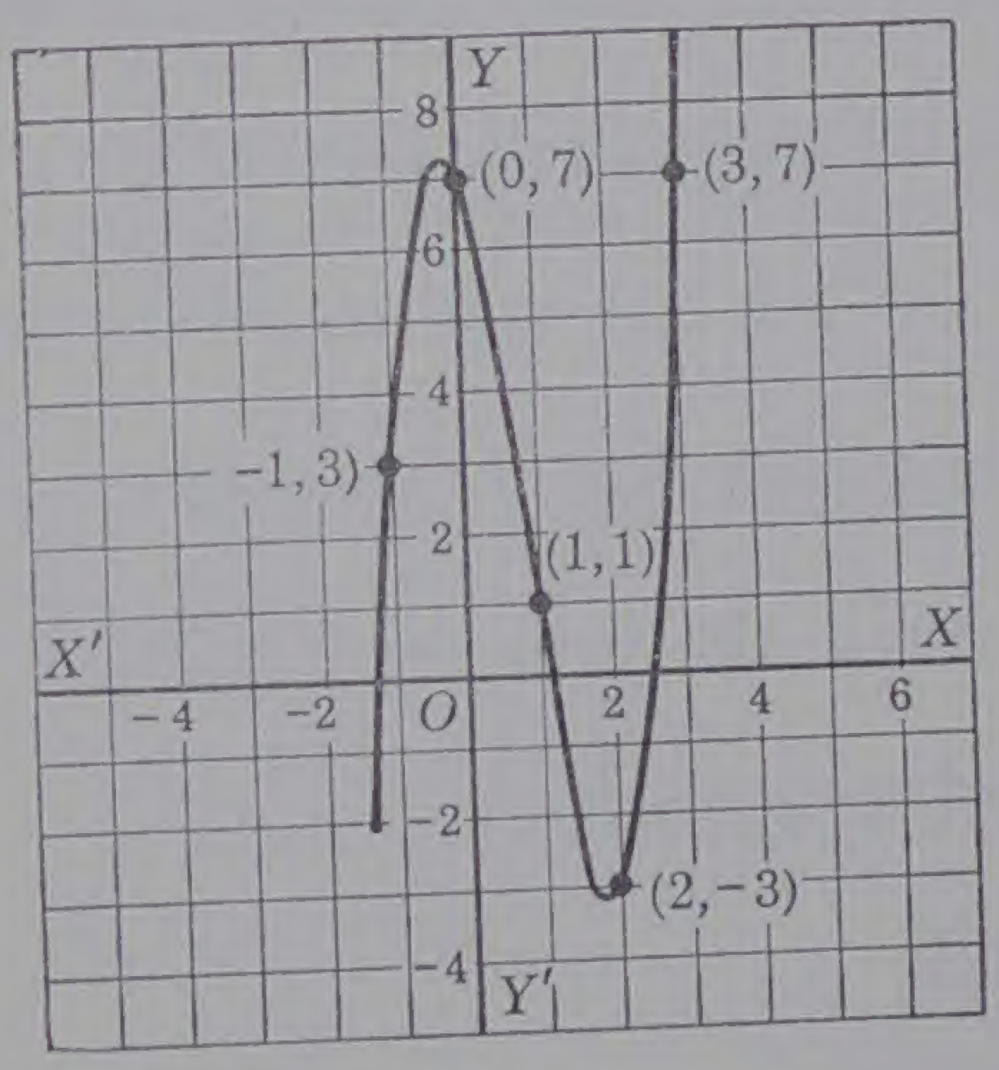


FIG. 30.

STEP 4. Apply transformation (3) to the first transformed equation to find a second transformed equation whose roots are $0.a_2$ units less than the roots of the first transformed equation.

STEP 5. The second transformed equation has a root between 0 and 0.1, say between $0.0a_3$ and $0.0a_3 + 0.01$. The approximate root of the second transformed equation is then $x_3 = 0.0a_3$.

This process may be continued until the irrational root is approximated to as many decimal places as may be desired. The approximate value of the root of the given equation is found by taking the sum of the successive approximations. Thus the approximate value of a given root is

$$\begin{aligned} x_1 &= a_1 \\ x_2 &= 0.a_2 \\ x_3 &= 0.0a_3 \\ \cdot\ \cdot\ \cdot &= \cdot\ \cdot\ \cdot \\ \hline x &= a_1.a_2a_3\ \cdot\ \cdot\ \cdot\ . \end{aligned}$$

This procedure is repeated for each of the irrational roots.

We shall illustrate Horner's method by solving the following example.

Example. Find all the irrational roots of the equation $f(x) = 2x^3 - 5x^2 - 3x + 7 = 0$.

Solution. Since $f(x)$ has two variations in sign, then by Descartes's rule of signs there are either two positive roots or no positive roots. From the equation

$$f(-x) = -2x^3 - 5x^2 + 3x + 7 = 0,$$

we see that $f(-x)$ has one variation in sign. Hence, the given equation must have *one negative* root.

We shall now locate the roots by synthetic division.

$$\begin{array}{r|l} 2-5-3+7 & 1 \\ \quad +2-3-6 & \\ \hline 2-3-6+1 & \end{array} \qquad \begin{array}{r|l} 2-5-3+7 & -1 \\ \quad -2+7-4 & \\ \hline 2-7+4+3 & \end{array}$$

$$\begin{array}{r|l} 2-5-3+\ 7 & 2 \\ \quad +4-2-10 & \\ \hline 2-1-5-\ 3 & \end{array} \qquad \begin{array}{r|l} 2-5-\ 3+\ 7 & -2 \\ \quad -4+18-30 & \\ \hline 2-9+15-23 & \end{array}$$

$$\begin{array}{r|l} 2-5-3+7 & 3 \\ \quad +6+3+0 & \\ \hline 2+1+0+7 & \end{array}$$

It follows from this that the given equation $f(x) = 0$ must have roots between 1 and 2, between 2 and 3, and between -1 and -2 because $f(1) = 1$ and $f(2) = -3$ are opposite in sign, $f(2) = -3$ and $f(3) = 7$ are opposite in sign, and $f(-1) = 3$ and $f(-2) = -23$ are opposite in sign. These results are recorded in the accompanying table, and the graph is given in Fig. 30. Additional points obtained from the work to follow have been included in the table.

x	y
0	7
1	1
1.1	0.312
1.14	0.045
1.15	−0.021
1.2	−0.344
2	−3
3	7
−1	3
−1.1	1.588
−1.19	0.019
−1.2	−0.056
−2	−23

From Fig. 30, *observe that if the graph of* $y = f(x)$ *goes down across the* x*-axis at a root of the equation* $f(x) = 0$, *then any value of* x *in the neighborhood of this root that gives a positive remainder by synthetic division is smaller than the root, and any value of* x *in the neighborhood of this root that gives a negative remainder by synthetic division is larger than the root.* However, *the opposite is true if the graph rises as it crosses the* x*-axis at a root.* The former case is illustrated in Fig. 30 at the root located between 1 and 2, and the latter at the roots located between 2 and 3 and between -1 and -2.

If a root is a double root, then all the remainders by synthetic division will be alike in sign for values of x in the neighborhood of this double root, whether x is less than the root or greater than the root.

Let us approximate the root between 1 and 2. Hence, $x_1 = a_1 = 1$.

Decreasing the roots of the given equation by 1 by the method described in Article 76, we have by synthetic division

$$\begin{array}{r|l} 2-5-3+7 & \underline{1} \\ \quad +2-3-6 & \\ \hline 2-3-6+\mathbf{1} & \\ \quad +2-1 & \\ \hline 2-1-\mathbf{7} & \\ \quad +2 & \\ \hline 2+\mathbf{1} & \end{array}$$

Therefore, 2, +1, −7, and +1 are the coefficients of the first transformed equation, and hence the first transformed equation is

$$2x^3 + x^2 - 7x + 1 = 0. \tag{I}$$

Since all the roots of Equation (I) are 1 unit less than the roots of the given equation, it follows that Equation (I) must have a root between 0 and 1. This root is located between 0.1 and 0.2 because the remainders are opposite in sign when we test 0.1 and 0.2.

Observe how the coefficients are written so as to take care of the proper number of decimal places.

$$\begin{array}{r|l} 2+1.0-7.00+1.000 & \underline{0.1} \\ \quad +0.2+0.12-0.688 & \\ \hline 2+1.2-6.88+\mathbf{0.312} & = f(1.1) \\ \quad +0.2+0.14 & \\ \hline 2+1.4-\mathbf{6.74} & \\ \quad 0.2 & \\ \hline 2+\mathbf{1.6} & \end{array}$$

$$\begin{array}{r|l} 2+1.0-7.00+1.000 & \underline{0.2} \\ \quad +0.4+0.28-1.344 & \\ \hline 2+1.4-6.72-0.344 & = f(1.2) \end{array}$$

Hence $x_2 = a_2 = 0.1$, and the second transformed equation is

$$2x^3 + 1.6x^2 - 6.74x + 0.312 = 0. \tag{II}$$

All the roots of Equation (II) are 0.1 of a unit less than the roots of Equation (I), and hence Equation (II) must have a root between 0 and 0.1. Observe that for small values of x the terms containing x^2 and higher powers of x will be small (provided the coefficients are not too large) in comparison with the term in x and the constant term. *Thus, an approximate root of any of the transformed equations is the solution of the linear part of these equations.* An approximation to the root between 0 and 0.1 of Equation (II) is the root of the equation $-6.74x + 0.312 = 0$, and

hence is

$$x = \frac{0.312}{6.74} = 0.047.$$

Testing 0.05, we have

$$\begin{array}{l} 2 + 1.60 - 6.7400 + 0.312000 \quad \lfloor 0.05 \\ \quad + 0.10 + 0.0850 - 0.332750 \\ \hline 2 + 1.70 - 6.6550 - 0.020750 = f(1.15) \end{array}$$

Since 0.05 gives a negative remainder, it is too large (see Fig. 30).

Testing 0.04, we have

$$\begin{array}{l} 2 + 1.60 - 6.7400 + 0.312000 \quad \lfloor 0.04 \\ \quad + 0.08 + 0.0672 - 0.266912 \\ \hline 2 + 1.68 - 6.6728 + \mathbf{0.045088} = f(1.14) \\ \quad + 0.08 + 0.0704 \\ \hline 2 + 1.76 - \mathbf{6.6024} \\ \quad + 0.08 \\ \hline \mathbf{2 + 1.84} \end{array}$$

Since 0.05 gives a negative remainder and 0.04 gives a positive remainder, Equation (II) must have a root between 0.04 and 0.05. Hence $x_3 = a_3 = 0.04$, and the third transformed equation is

$$2x^3 + 1.84x^2 - 6.6024x + 0.045088 = 0. \tag{III}$$

Equation (III) must have a root between 0 and 0.01. An approximation to this root is the root of the equation $-6.6024x + 0.045088 = 0$, and hence is

$$x = \frac{0.045088}{6.6024} = 0.0068.$$

Thus $x_4 = a_4 = 0.007$.

Therefore, a root of the given equation correct to three decimal places is

$$\begin{array}{rl} x_1 = a_1 & = 1.000 \\ x_2 = a_2 & = 0.100 \\ x_3 = a_3 & = 0.040 \\ x_4 = a_4 & = 0.007 \\ \hline x = & 1.147 \end{array}$$

As a matter of fact, this root correct to four decimal places is $x = 1.1468$.

By continuing in this manner, it is possible to find the irrational roots of an algebraic equation correctly to as many decimal places as may be desired.

We shall now approximate the root between 2 and 3 of the given equation $2x^3 - 5x^2 - 3x + 7 = 0$.

Let $x_1 = 2$ and divide:

$$\begin{array}{r|l} 2 - 5 - 3 + 7 & 2 \\ \quad + 4 - 2 - 10 & \\ \hline 2 - 1 - 5 - \mathbf{3} & \\ \quad + 4 + 6 & \\ \hline 2 + 3 + \mathbf{1} & \\ \quad + 4 & \\ \hline 2 + \mathbf{7} & \end{array}$$

Hence the first transformed equation is

$$2x^3 + 7x^2 + x - 3 = 0. \tag{I}$$

Let $x_2 = 0.6$ and divide:

$$\begin{array}{r|l} 2 + 7.0 + 1.00 - 3.000 & 0.6 \\ \quad + 1.2 + 4.92 + 3.552 & \\ \hline 2 + 8.2 + 5.92 + \mathbf{0.552} & \\ \quad + 1.2 + 5.64 & \\ \hline 2 + 9.4 + \mathbf{11.56} & \\ \quad + 1.2 & \\ \hline 2 + \mathbf{10.6} & \end{array}$$

therefore the second transformed equation is

$$2x^3 + 10.6x^2 + 11.56x + 0.552 = 0. \tag{II}$$

Then the approximate root may be obtained by the solution of

$$11.56x + 0.552 = 0,$$

and therefore

$$x_3 = -\frac{0.552}{11.56} = -0.05.$$

It is possible to approximate irrational roots by using numbers either a little greater or a little less than the given root.

Let $x_3 = -0.05$ and divide:

$$\begin{array}{r|l} 2 + 10.60 + 11.5600 + 0.552000 & -0.05 \\ \quad - 0.10 - 0.5250 - 0.551750 & \\ \hline 2 + 10.50 + 11.0350 + \mathbf{0.000250} & \\ \quad - 0.10 - 0.5200 & \\ \hline 2 + 10.40 + \mathbf{10.5150} & \\ \quad - 0.10 & \\ \hline 2 + \mathbf{10.30} & \end{array}$$

Therefore the third transformed equation is

$$2x^3 + 10.30x^2 + 10.5150x + 0.000250 = 0. \tag{III}$$

The approximate root is obtained by the solution of

$$10.5150x + 0.000250 = 0,$$

and therefore

$$x_4 = -\frac{0.000250}{10.5150}$$
$$= -0.0000238.$$

Therefore the root is

$$\begin{array}{rl} x_1 = & 2.00000 \\ x_2 = & 0.60000 \\ x_3 = & -0.05000 \\ x_4 = & -0.0000238 \\ \hline x = & 2.5499762 \end{array}$$

The negative root of the given equation can be approximated in the following manner. The given equation is

$$2x^3 - 5x^2 - 3x + 7 = 0.$$

Testing -1 and -2, we have

$$\begin{array}{rrrrl} 2 & - 5 & - 3 & + 7 & \vert -1 \\ & - 2 & + 7 & - 4 & \\ \hline 2 & - 7 & + 4 & + \mathbf{3} & \\ & - 2 & + 9 & & \\ \hline 2 & - 9 & + \mathbf{13} & & \\ & - 2 & & & \\ \hline 2 & - \mathbf{11} & & & \end{array}$$

$$\begin{array}{rrrrl} 2 & - 5 & - 3 & + 7 & \vert -2 \\ & - 4 & + 18 & - 30 & \\ \hline 2 & - 9 & + 15 & - 23 & \end{array}$$

Let $x_1 = a_1 = -1$; then the first transformed equation is

$$2x^3 - 11x^2 + 13x + 3 = 0. \tag{I}$$

Hence the approximate root is found by the solution of

$$13x + 3 = 0,$$
$$x = -\frac{3}{13} = -0.23.$$

Testing -0.1 and -0.2 we have

$$\begin{array}{rrrrl} 2 & - 11.0 & + 13.00 & + 3.000 & \vert -0.1 \\ & - 0.2 & + 1.12 & - 1.412 & \\ \hline 2 & - 11.2 & + 14.12 & + \mathbf{1.588} & = f(-1.1) \\ & - 0.2 & + 1.14 & & \\ \hline 2 & - 11.4 & + \mathbf{15.26} & & \\ & - 0.2 & & & \\ \hline 2 & - \mathbf{11.6} & & & \end{array}$$

$$\begin{array}{rrrrl} 2 & - 11.0 & + 13.00 & + 3.000 & \vert -0.2 \\ & - 0.4 & + 2.28 & - 3.056 & \\ \hline 2 & - 11.4 & + 15.28 & - 0.056 & = f(-1.2) \end{array}$$

Since the remainder is positive for $x = -0.1$ and negative for $x = -0.2$, the root of the first transformed equation lies between -0.1 and -0.2. Let $x_2 = a_2 = -0.1$; then the second transformed equation is

$$2x^3 - 11.6x^2 + 15.26x + 1.588 = 0. \tag{II}$$

For the approximate root solve:

$$15.26x + 1.588 = 0.$$

$$x = -\frac{1.588}{15.26} = -0.104.$$

Testing -0.09, we have

$$\begin{array}{r|l} 2 - 11.60 + 15.2600 + 1.588000 & -0.09 \\ \quad - \ 0.18 + \ 1.0602 - 1.468818 & \\ \hline 2 - 11.78 + 16.3202 + \mathbf{0.119182} & = f(-1.19) \\ \quad - \ 0.18 + \ 1.0764 & \\ \hline 2 - 11.96 + \mathbf{17.3966} & \\ \quad - \ 0.18 & \\ \hline \mathbf{2 - 12.14} & \end{array}$$

Let $x_3 = a_3 = -0.09$, then the third transformed equation is

$$2x^3 - 12.14x^2 + 17.3966x + 0.119182 = 0. \tag{III}$$

For the approximate root solve:

$$17.3966x + 0.119182 = 0;$$

$$x = -\frac{0.119182}{17.3966} = -0.0068.$$

Hence $x_4 = a_4 = -0.007$.

Therefore, the negative root of the given equation correct to three decimal places is

$$\begin{array}{rl} x_1 = a_1 = & -1.000 \\ x_2 = a_2 = & -0.100 \\ x_3 = a_3 = & -0.090 \\ x_4 = a_4 = & -0.007 \\ \hline x \qquad\quad = & -1.197. \end{array}$$

Thus, the three roots of the equation $2x^3 - 5x^2 - 3x + 7 = 0$ are 1.147, 2.550, and -1.197.

The negative roots of any equation $f(x) = 0$ *can be approximated by approximating the positive roots of the equation* $f(-x) = 0$.

Straight-Line Method

If the graph of $y = f(x)$ is approximately a straight line in the neighborhood of a real root of the equation $f(x) = 0$, then a quick

approximation for this root, whether rational or irrational, can be obtained by *the straight-line method*. The straight-line method consists of choosing two points on the curve, the abscissa of the one point being a little less than the root and the abscissa of the other point being a little greater than the root; then an approximation to a root is the x-intercept of the straight line drawn through these two points.

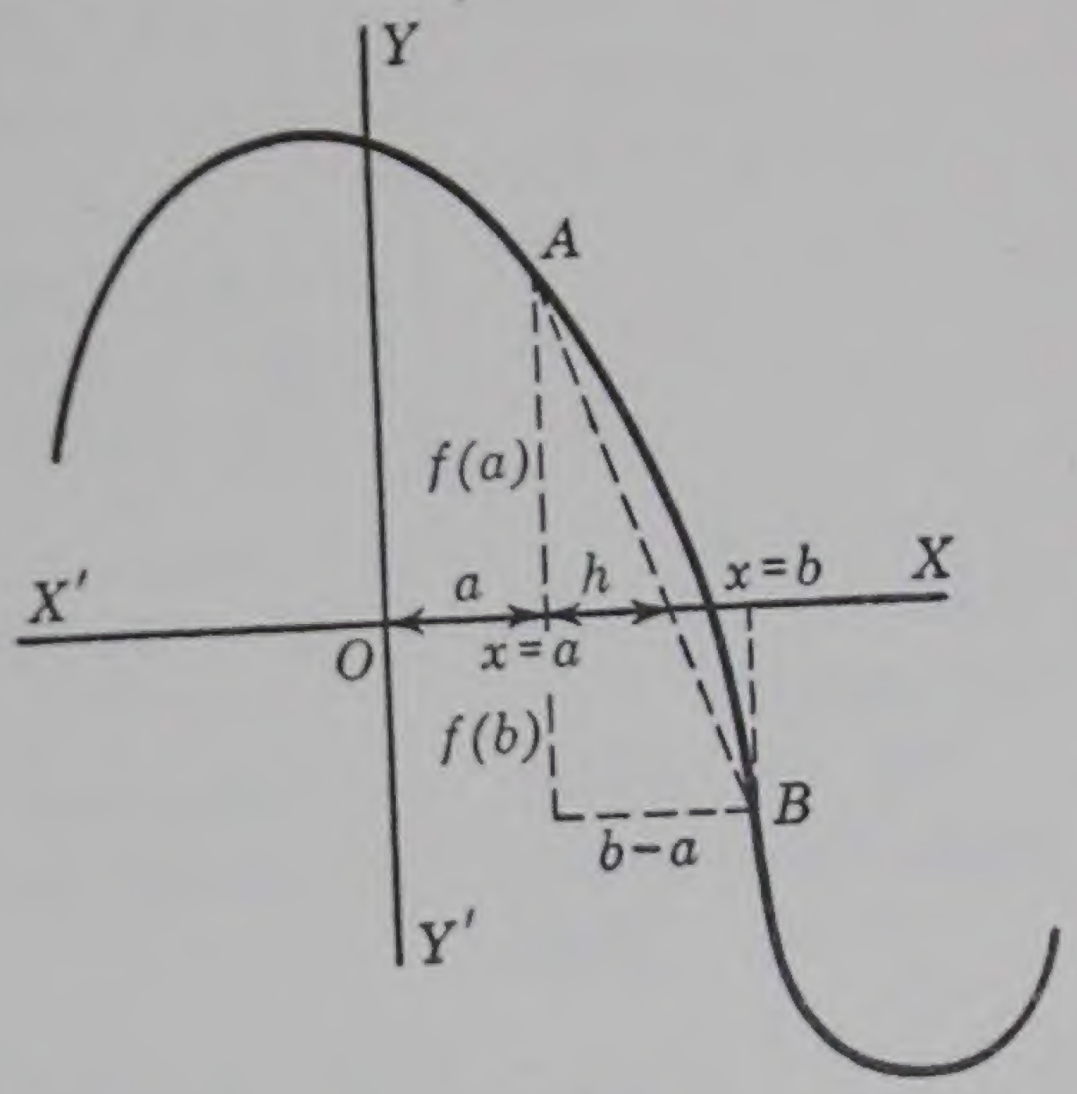


FIG. 31.

In Fig. 31, connect the points A and B on the graph of $y = f(x)$ by a straight line. Let a be the abscissa of the point A, and b the abscissa of the point B, and let h be the distance from $x = a$ to the point where the straight line crosses the x-axis. An approximation to the root of the equation $f(x) = 0$ is $x = a + h$. From similar triangles

$$\frac{h}{b-a} = \frac{f(a)}{f(a) - f(b)}; \tag{32}$$

solving for h

$$h = \frac{(b-a)f(a)}{f(a) - f(b)}. \tag{33}$$

Since $f(a)$ and $f(b)$ are opposite in sign, the sum of their ordinates is given by $f(a) - f(b)$.

We shall illustrate the straight-line method for approximating the roots by approximating the root which lies between 1 and 2 of the equation $2x^3 - 5x^2 - 3x + 7 = 0$.

Solution. Here we let $a = 1$ and $b = 2$, and divide as follows:

$$\begin{array}{r|l} 2-5-3+7 & 1 \\ \quad +2-3-6 & \\ \hline 2-3-6+1 & = f(1) \end{array} \qquad \begin{array}{r|l} 2-5-3+\ 7 & 2 \\ \quad +4-2-10 & \\ \hline 2-1-5-\ 3 & = f(2) \end{array}$$

Then $f(a) = f(1) = 1$ and $f(b) = f(2) = -3$. Hence, by Equation (33)

$$h_1 = \frac{(b-a)f(a)}{f(a) - f(b)} = \frac{(2-1)(1)}{(1) - (-3)} = \frac{1}{4} = 0.25.$$

Therefore an approximation to the root is $x = a + h_1 = 1 + 0.25 = 1.25$.

Let us test to see whether the root is greater than or less than 1.2. Testing 1.2 we have

$$\begin{array}{r|l} 2 - 5.0 - 3.00 + 7.000 & 1.2 \\ \quad + 2.4 - 3.12 - 7.344 & \\ \hline 2 - 2.6 - 6.12 - 0.344 = f(1.2) & \end{array}$$

Testing 1.1 we have

$$\begin{array}{r|l} 2 - 5.0 - 3.00 + 7.000 & 1.1 \\ \quad + 2.2 - 3.08 - 6.688 & \\ \hline 2 - 2.8 - 6.08 + 0.312 = f(1.1) & \end{array}$$

Since $f(1.2) = -0.344$ is negative, we see from Fig. 30 that the root is less than 1.2. Since $f(1.1) = 0.312$ is positive, the root must be between 1.1 and 1.2.

For the second approximation we let $a = 1.1$ and $b = 1.2$, then from the foregoing calculations $f(a) = f(1.1) = 0.312$ and $f(b) = f(1.2) = -0.344$. Hence,

$$h_2 = \frac{(1.2 - 1.1)(0.312)}{(0.312) - (-0.344)} = \frac{0.0312}{0.656} = 0.047.$$

Therefore, a closer approximation to the root is $x = a + h_2 = 1.1 + 0.047 = 1.147$. The process may be continued if a greater degree of accuracy is desired. (See approximation by Horner's method, page 263.)

Newton's Method

In 1669 Sir Isaac Newton (1642–1727) published a method for approximating the irrational roots of an equation $f(x) = 0$. In principle the method consists of drawing a tangent line to the curve $y = f(x)$ at a point whose abscissa is approximately a root of the given equation; then the abscissa of the point where this tangent line crosses the x-axis is an approximation to the root. *If there are no turning points nor points of inflection in the neighborhood of the root, then the tangent line will cross the x-axis at a point approximately where the curve of* $y = f(x)$ *crosses the x-axis.* This process can be continued by drawing a second tangent line at the point on the curve whose abscissa is the x-intercept of the first tangent line, and so on.

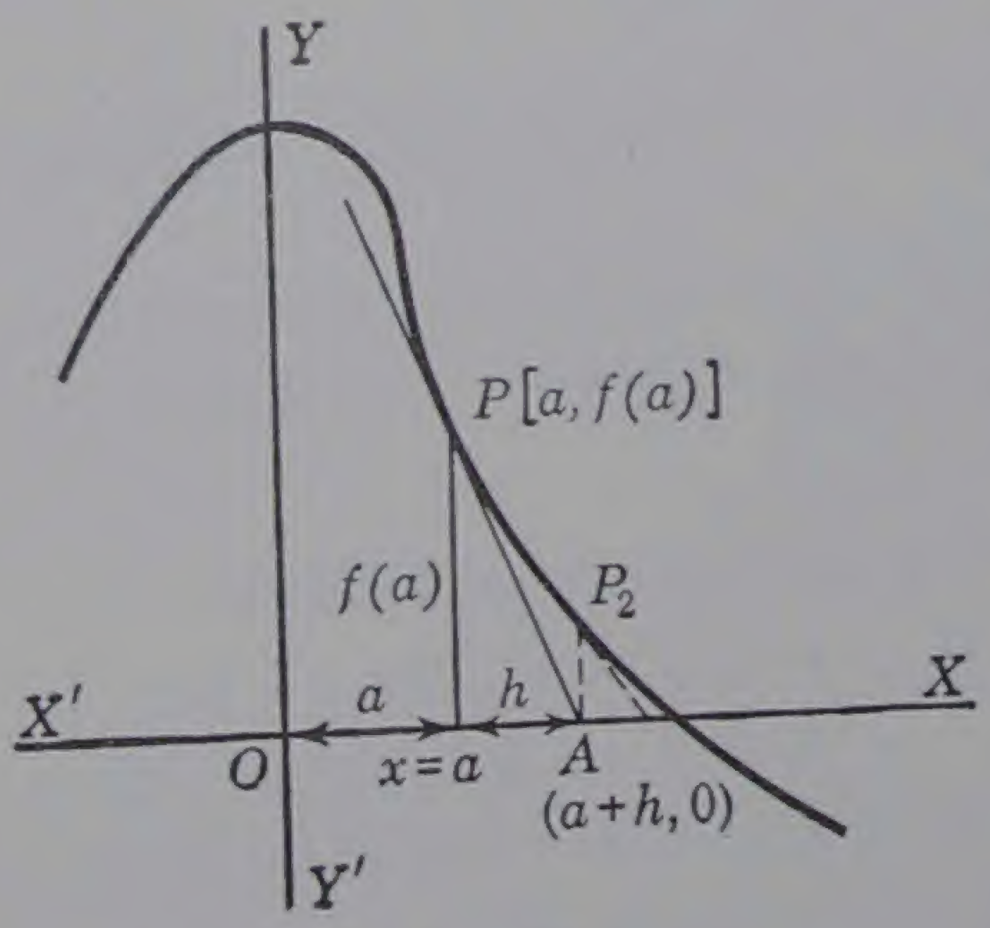


FIG. 32.

For the general case, suppose that the tangent line is drawn at the point P on the curve of $f(x)$ where $x = a$ (see Fig. 32). The ordinate of this point is $f(a)$, and the coördinates of the point P are therefore $[a, f(a)]$. Let h be the distance from $x = a$ to the point A where the tangent line crosses the x-axis. Therefore, the coördinates of the point A are $(a + h, 0)$.

By Article 41 the slope of the tangent line is $f'(a)$, and by Equation (12), Article 41, the slope is $\dfrac{f(a) - 0}{a - (a + h)}$. Therefore

$$f'(a) = \frac{f(a) - 0}{a - (a + h)} = \frac{f(a)}{-h}. \tag{34}$$

Solving for h,

$$h = -\frac{f(a)}{f'(a)}. \tag{35}$$

If a is less than the root, the slope of the tangent line is negative when $f(a)$ is positive, and positive when $f(a)$ is negative, hence h is positive. If a is greater than the root, $f(a)$ and $f'(a)$ are alike in sign, hence h is negative.

We shall illustrate Newton's method by approximating the root that lies between 2 and 3 of the equation $2x^3 - 5x^2 - 3x + 7 = 0$.

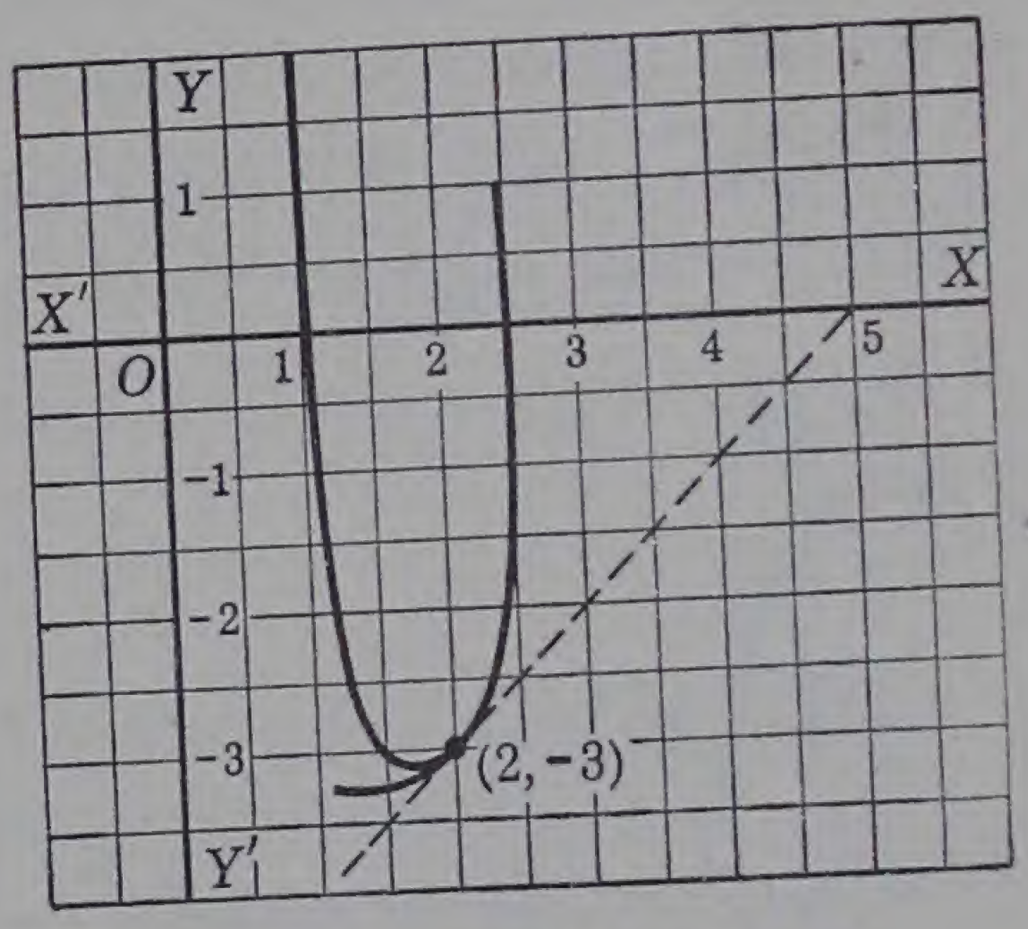


FIG. 33.

Solution. From Fig. 33, we see that the graph of $y = f(x)$ has a turning point in the neighborhood of $x = 2$. Thus we should not let $a = 2$ but should let $a = 3$ instead.

The given equation is

$$f(x) = 2x^3 - 5x^2 - 3x + 7 = 0$$

and the derivative is

$$f'(x) = 6x^2 - 10x - 3.$$

Let $x_1 = a_1 = 3$, then

$$h_1 - \frac{f(a_1)}{f'(a_1)} = -\frac{f(3)}{f'(3)} = -\frac{7}{21} = -0.33.$$

Therefore $x_2 = a_1 + h_1 = 3 - 0.33 = 2.67$.

Let $a_2 = 2.6$, then

$$h_2 = -\frac{f(2.6)}{f'(2.6)} = -\frac{0.552}{11.56} = -0.048.$$

Therefore $x_3 = a_2 + h_2 = 2.6 - 0.048 = 2.552$.

Let $a_3 = 2.55$, then

$$h_3 = -\frac{f(2.55)}{f'(2.55)} = -\frac{0.00025}{10.515} = -0.0000238.$$

Therefore $x_4 = a_3 + h_3 = 2.55 - 0.0000238 = 2.5499762$. (See solution by Horner's method, page 265.)

If we had let $x_1 = a = 2$, the calculations would be as follows:

$$h_1 = -\frac{f(a)}{f'(a)} = -\frac{f(2)}{f'(2)} = -\frac{-3}{1} = 3.$$

Hence the second approximation of the root would be $x_2 = a + h_1 = 2 + 3 = 5$, which is not a very close approximation to the root between 2 and 3. However, if we were to continue the process we would eventually approximate the root between 2 and 3.

Newton's method is more general than Horner's method. Both algebraic and transcendental equations can be solved by Newton's method while only algebraic equations can be solved by Horner's method.

Newton's method is sometimes considered to be shorter than Horner's method because an irrational root can be determined correctly to several additional decimal places with each operation. However, it is possible to use more than one decimal at a time with Horner's method. For example, when approximating the root between 1 and 2 we could have used 0.047 instead of 0.04 when transforming Equation (II) under Horner's method. (See example in Article 77.)

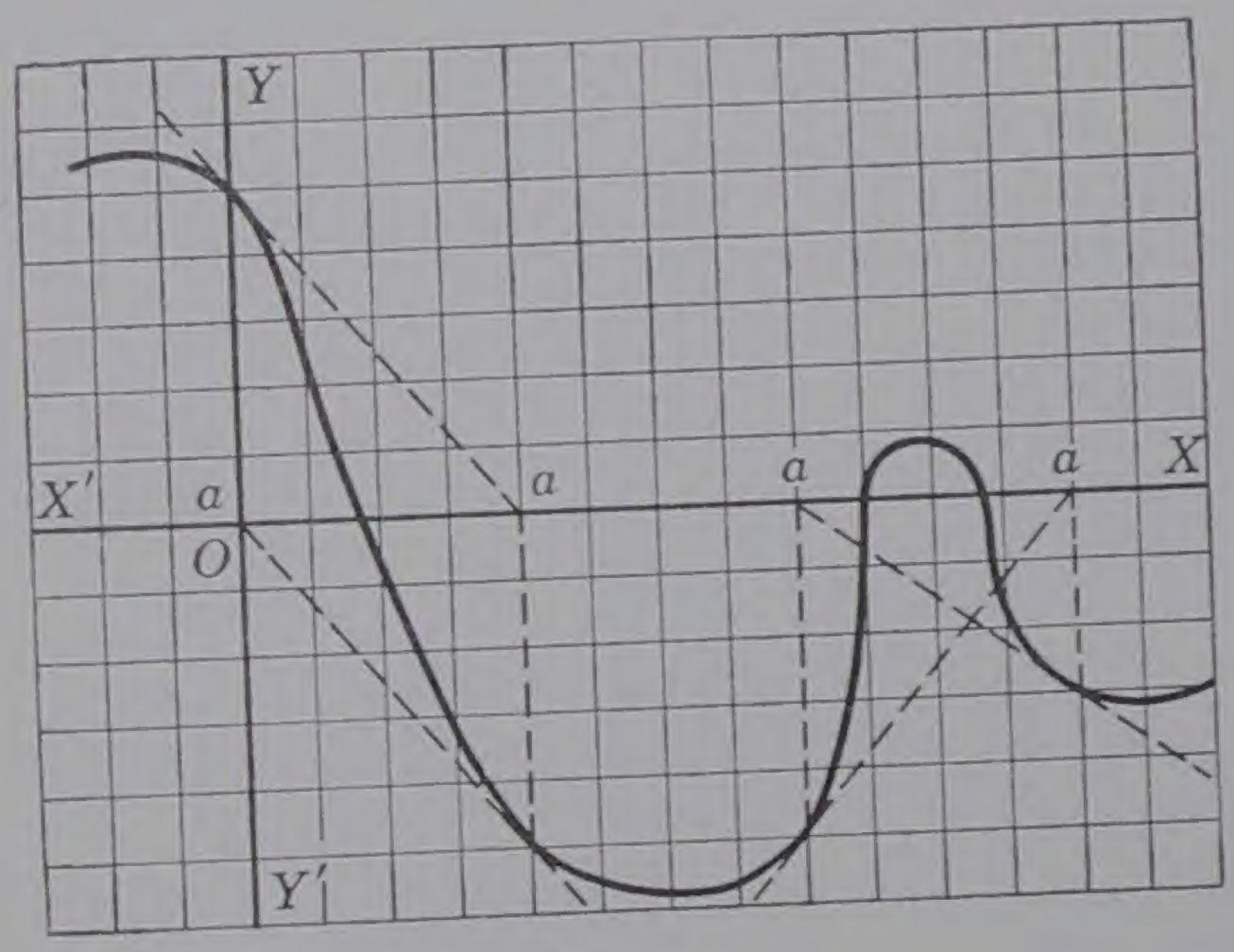


FIG. 34.

Care must be exercised when using Newton's method *not to choose for an approximate root a value for* a *if the graph of* $y = f(x)$ *has either a maximum or a minimum point in the neighborhood of*

$x = a$, or if the graph has a point of inflection in the neighborhood of $x = a$. Fig. 33 and Fig. 34 illustrate why care is necessary when selecting an approximate root. Fig. 34 illustrates how Newton's method may fail to approximate any root if a is chosen as indicated.

An irrational root can be approximated by any of the three methods decribed, provided the root has first been located, usually between two consecutive integers.* Under certain conditions we may have difficulty in locating some of the irrational roots. This is especially true if some of the roots are repeated, or if two or more irrational roots lie between two consecutive integers. In Article 41 we saw how to find the maximum and minimum values of a function $f(x)$. This information is very useful in determining whether or not the graph of $y = f(x)$ crosses the x-axis two or more times between two consecutive integers. In this way it is possible to determine whether or not the equation $f(x) = 0$ has two or more roots between any two consecutive integers.

A sure test for determining the exact number and the location of all the real roots of a numerical equation was published in 1829 by a Swiss mathematician J. C. F. Sturm (1803–1855), and is known as Sturm's theorem. This method is given in many treatises on the theory of equations.

EXERCISES

Find all the real roots of each of the equations (Exercises 1 through 32). Give the irrational roots correct to three decimal places.

1. $x^3 - 5x^2 + 3x + 6 = 0.$
2. $x^3 + 2x^2 + 3x + 4 = 0.$
3. $x^3 - 6 = 0.$
4. $x^3 + 9 = 0.$
5. $x^3 - 6x^2 + 3x + 5 = 0.$
6. $3x^3 - 5x^2 - 9 = 0.$
7. $4x^3 - 33x - 20 = 0.$
8. $x^4 - 3x^3 + 4x - 12 = 0.$
9. $x^3 - 7x + 3 = 0.$
10. $5x^3 - 3x^2 - 7x + 8 = 0.$
11. $x^3 + 3x^2 - 9x - 5 = 0.$
12. $x^3 - 3x - 1 = 0.$
13. $x^3 + 2x^2 + 3x + 4 = 0.$
14. $4x^3 - 17x^2 + 24x - 15 = 0.$
15. $2x^4 - x^3 - 10x^2 + 2x + 12 = 0.$
16. $16x^3 - 10x^2 + 23x - 36 = 0.$
17. $3x^5 - 7x^4 - 6x^3 + 9x + 6 = 0.$
18. $x^3 - 6x^2 + 3x + 5 = 0.$
19. $4x^3 - 15x^2 + 30 = 0.$
20. $6x^3 - 15x^2 + 13 = 0.$
21. $x^3 - 5x^2 + 2 = 0.$

* In the case of Newton's method, it is necessary for the root to be located in an interval meeting all the necessary conditions.

22. $x^4 + 4x^3 - 2x^2 + 32 = 0.$
23. $x^3 - 2x^2 - 13x + 18 = 0.$
24. $x^4 + 2x^3 + x^2 - x - 5 = 0.$
25. $x^3 - 3x^2 - 2x + 5 = 0.$
26. $3x^3 - 2x^2 + x - 1 = 0.$
27. $2x^3 - 3x^2 - 12x + 21 = 0.$
28. $3x^3 - 2x^2 + x + 1 = 0.$
29. $4x^4 - 3x^3 - 6x^2 + 5 = 0.$
30. $2x^4 - x^2 - 6 = 0.$
31. $16x^3 - 18x^2 + 24x - 27 = 0.$
32. $3x^4 - 5x^2 + 6x - 8 = 0.$

33. A sphere of radius r inches and having a specific gravity s will sink to a depth of x inches given by the equation $x^3 - 3rx^2 + 4r^3s = 0$. (a) For ice $s = 0.92$, find the depth for a sphere 10 inches in radius. (b) For pine $s = 0.5$, find the depth for a sphere 2 feet in diameter. (c) For oak $s = 0.8$, find the depth for a sphere 6 inches in radius. (d) What size oak sphere will sink 4 inches?

34. A buoy made in the form of a hollow iron sphere has an external radius of 3 feet. What must be the thickness of the wall if the sphere will just float in water? Iron weighs 7.86 times as much as water.

35. One ladder has its base against house A and its top against house B; the other ladder has its base against house B and its top against house A. The two ladders cross 8 feet from the ground. The ladders are 12 and 16 feet long, respectively. How far apart are the houses?

78. SOLUTION OF THE CUBIC EQUATION

To solve the general cubic equation

$$ax^3 + bx^2 + cx + d = 0, \tag{36}$$

we transform it into the form of the equation

$$y^3 + py + q = 0. \tag{37}$$

This is accomplished by setting

$$x = y - \frac{b}{3a} \tag{38}$$

in Equation (36), a transformation that increases the roots by $\frac{b}{3a}$. Hence,

$$a\left(y - \frac{b}{3a}\right)^3 + b\left(y - \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d = 0;$$

simplifying gives

$$y^3 + \frac{3ac - b^2}{3a^2}\, y + \frac{2b^3 - 9abc + 27a^2d}{27a^3} = 0. \quad (39)$$

This equation reduces to $y^3 + py + q = 0$ if we set

$$p = \frac{3ac - b^2}{3a^2}, \quad (40)$$

and

$$q = \frac{2b^3 - 9abc + 27a^2d}{27a^3}. \quad (41)$$

To solve the equation $y^3 + py + q = 0$, we introduce two unknowns, u and v. Since there are two unknowns we can impose two conditions upon these unknowns. For the first condition we let their sum be a root of this cubic equation. Thus we set

$$u + v = y. \quad (42)$$

Substituting in Equation (37) we get

$$(u + v)^3 + p(u + v) + q = 0; \quad (43)$$

simplifying, gives

$$u^3 + v^3 + (3uv + p)(u + v) + q = 0. \quad (44)$$

For the second condition we set

$$(3uv + p) = 0 \qquad \text{or} \qquad v = -\frac{p}{3u}. \quad (45)$$

Substituting in Equation (44), we have

$$u^3 + \left(-\frac{p}{3u}\right)^3 + q = 0;$$

simplifying, gives

$$u^6 + qu^3 - \frac{p^3}{27} = 0. \quad (46)$$

This equation is a quadratic in u^3, and hence can be solved for u^3 by the quadratic formula

$$u^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}. \quad (47)$$

From Equations (44) and (45)

$$v^3 = -q - u^3;$$

hence

$$v^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}. \quad (48)$$

The double sign before each radical is omitted because the other choice of signs would simply interchange the values of u and v.

Let u_1 and v_1 be the principal cube roots of the right-hand member of Equations (47) and (48), respectively; that is, let

$$u_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}, \tag{49}$$

and

$$v_1 = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}. \tag{50}$$

By Example 1, Article 59, the three cube roots of 1 are 1, ω, and ω^2. Hence the three values of u are u_1, ωu_1, $\omega^2 u_1$ and the three values of v are v_1, ωv_1, $\omega^2 v_1$, where $\omega = -\frac{1}{2} + i\frac{1}{2}\sqrt{3}$ and $\omega^2 = -\frac{1}{2} - i\frac{1}{2}\sqrt{3}$.

By Equation (45) $uv = -\frac{p}{3}$, thus the values of u and v must be paired in such a manner that their products are equal. The only pairs of values that satisfy this condition are

$$u_1 \text{ and } v_1; \qquad \omega u_1 \text{ and } \omega^2 v_1; \qquad \omega^2 u_1 \text{ and } \omega v_1.$$

Therefore, the three roots of Equation (37) are

$$\begin{aligned} y_1 &= u_1 + v_1, \\ y_2 &= \omega u_1 + \omega^2 v_1, \\ y_3 &= \omega^2 u_1 + \omega v_1. \end{aligned} \tag{51}$$

By means of Equations (45) and (38), the roots of the general cubic Equation (36) are

$$\begin{aligned} x_1 &= u_1 - \frac{p}{3u_1} - \frac{b}{3a}, \\ x_2 &= \omega u_1 - \frac{p}{3\omega u_1} - \frac{b}{3a}, \\ x_3 &= \omega^2 u_1 - \frac{p}{3\omega^2 u_1} - \frac{b}{3a}. \end{aligned} \tag{52}$$

If the coefficients of the given cubic equation are real numbers, the equation has one real root and two imaginary roots if $\frac{q^2}{4} + \frac{p^3}{27}$ is positive, and two equal roots if $\frac{q^2}{4} + \frac{p^3}{27}$ is zero.

If $\frac{q^2}{4} + \frac{p^3}{27}$ is negative, the three roots are real and distinct. However, to obtain u in this case, by means of Equation (49), would involve the extraction of the cube root of imaginary numbers algebraically. Since no such algebraic process exists, this case is called the **irreducible case.** However, the three real roots can be obtained by a method involving trigonometry. This method will be discussed in Article 79.

Example. Solve $2x^3 + x^2 - 2x - 6 = 0$.

Solution. In this case $a = 2, b = 1, c = -2, d = -6$.

$$p = \frac{3ac - b^2}{3a^2} = \frac{-12 - 1}{12} = -\frac{13}{12},$$

$$q = \frac{2b^3 - 9abc + 27a^2d}{27a^3} = \frac{2 + 36 - 648}{216} = -\frac{305}{108},$$

$$u_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} = \sqrt[3]{+\frac{305}{216} + \sqrt{\frac{3 \cdot 29^2}{6^4}}}$$

$$= \tfrac{1}{6}\sqrt[3]{305 + 174\sqrt{3}} = \tfrac{1}{6}(5 + 2\sqrt{3}).$$

Since

$$x_1 = u_1 - \frac{p}{3u_1} - \frac{b}{3a},$$

then

$$x_1 = \frac{1}{6}(5 + 2\sqrt{3}) + \frac{13}{6(5 + 2\sqrt{3})} - \frac{1}{6}$$

$$= \tfrac{1}{6}[5 + 2\sqrt{3} + (5 - 2\sqrt{3}) - 1]$$

$$= \tfrac{3}{2}.$$

The other two roots can be obtained from Equation (52), and they are $x_2 = -1 + i$ and $x_3 = -1 - i$.

79. IRREDUCIBLE CASE

If $\frac{q^2}{4} + \frac{p^3}{27}$ is negative for the equation $y^3 + py + q = 0$, then by Equation (47) u^3 is imaginary. We can express this imaginary number in the trigonometric form

$$u^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = r(\cos\theta + i\sin\theta), \tag{53}$$

and the conjugate imaginary number v^3 in the form

$$v^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = r(\cos\theta - i\sin\theta). \tag{54}$$

Adding corresponding members of Equations (53) and (54), we obtain

$$-q = 2r \cos \theta; \tag{55}$$

multiplying corresponding members of Equations (53) and (54) we obtain

$$\frac{q^2}{4} - \left(\frac{q^2}{4} + \frac{p^3}{27}\right) = r^2(\cos^2 \theta + \sin^2 \theta) = r^2. \tag{56}$$

Solving Equations (56) and (55) for r and $\cos \theta$, we obtain

$$r = \sqrt{-\frac{p^3}{27}}, \tag{57}$$

and

$$\cos \theta = -\frac{q}{2r}. \tag{58}$$

By means of de Moivre's theorem (Article 59), if

$$u^3 = r(\cos \theta + i \sin \theta), \qquad \text{Equation (53)},$$

then

$$\begin{aligned} u_1 &= \sqrt[3]{r}\left[\cos \frac{\theta}{3} + i \sin \frac{\theta}{3}\right], \\ u_2 &= \sqrt[3]{r}\left[\cos\left(\frac{\theta}{3} + 120^\circ\right) + i \sin\left(\frac{\theta}{3} + 120^\circ\right)\right], \\ u_3 &= \sqrt[3]{r}\left[\cos\left(\frac{\theta}{3} + 240^\circ\right) + i \sin\left(\frac{\theta}{3} + 240^\circ\right)\right]. \end{aligned} \tag{59}$$

And if

$$v^3 = r(\cos \theta - i \sin \theta), \qquad \text{Equation (54)},$$

then

$$\begin{aligned} v_1 &= \sqrt[3]{r}\left[\cos \frac{\theta}{3} - i \sin \frac{\theta}{3}\right], \\ v_2 &= \sqrt[3]{r}\left[\cos\left(\frac{\theta}{3} + 120^\circ\right) - i \sin\left(\frac{\theta}{3} + 120^\circ\right)\right], \\ v_3 &= \sqrt[3]{r}\left[\cos\left(\frac{\theta}{3} + 240^\circ\right) - i \sin\left(\frac{\theta}{3} + 240^\circ\right)\right]. \end{aligned} \tag{60}$$

By Equation (45), $uv = -\frac{p}{3}$. The only pairs of values of u and v that satisfy this condition are

$$u_1 \text{ and } v_1; \qquad u_2 \text{ and } v_2; \qquad u_3 \text{ and } v_3.$$

Since $y = u + v$, and $x = y - \frac{b}{3a}$, the three roots of the general cubic Equation (36) are

$$\begin{aligned} x_1 &= 2\sqrt[3]{r}\cos\frac{\theta}{3} - \frac{b}{3a}, \\ x_2 &= 2\sqrt[3]{r}\cos\left(\frac{\theta}{3} + 120^\circ\right) - \frac{b}{3a}, \\ x_3 &= 2\sqrt[3]{r}\cos\left(\frac{\theta}{3} + 240^\circ\right) - \frac{b}{3a}. \end{aligned} \tag{61}$$

By Equation (57), r is imaginary if p is positive. Therefore, this method should be used only if p is negative. If p is negative, by Equation (61) all the roots are real. That is, this is a convenient method for solving cubic equations when all three of the roots are real.

To illustrate the irreducible case, let us solve the example in Article 77.

Example. Solve $2x^3 - 5x^2 - 3x + 7 = 0$.

Solution. In this case, $a = 2$, $b = -5$, $c = -3$, $d = 7$.

$$p = \frac{-18 - 25}{12} = -\frac{43}{12}.$$

$$q = \frac{-250 - 270 + 756}{216} = \frac{59}{54}.$$

$$r = \sqrt{-\frac{p^3}{27}} = \sqrt{\frac{79{,}507}{46{,}656}}.$$

$$\cos\theta = -\frac{q}{2r} = -0.4184845.$$

$$\theta = 114^\circ\ 44'\ 20''.2,$$

$$\frac{\theta}{3} = 38^\circ\ 14'\ 46''.7.$$

$$2\sqrt[3]{r} = 2.185813.$$

$$\begin{aligned} x_1 &= 2\sqrt[3]{r}(\cos 38^\circ\ 14'\ 46''.7) + \tfrac{5}{6} = 1.716643 + \tfrac{5}{6} = 2.549976, \\ x_2 &= 2\sqrt[3]{r}(\cos 158^\circ\ 14'\ 46''.7) + \tfrac{5}{6} = -2.030151 + \tfrac{5}{6} = -1.196818, \\ x_3 &= 2\sqrt[3]{r}(\cos 278^\circ\ 14'\ 46''.7) + \tfrac{5}{6} = 0.313508 + \tfrac{5}{6} = 1.146842. \end{aligned}$$

Compare these results with the results obtained in Article 77.

EXERCISES

Solve each of the following equations (Exercise 1 through 20).

1. $x^3 - 9x - 12 = 0$.

2. $x^3 + 3x^2 - 6x - 20 = 0$.

3. $x^3 - 3x^2 - 6x - 4 = 0$.

4. $x^3 - 6x^2 + 3x - 2 = 0$.

5. $x^3 + 2x^2 + 3x + 4 = 0.$
6. $3x^3 - 5x^2 - 9 = 0.$
7. $5x^3 - 3x^2 - 7x + 8 = 0.$
8. $3x^3 - 2x^2 + x + 1 = 0.$
9. $x^3 - 12x + 16 = 0.$
10. $x^3 - 3x - 2 = 0.$
11. $x^3 + 3x^2 - 9x + 5 = 0.$
12. $2x^3 + x^2 - 4x - 3 = 0.$
13. $x^3 - 7x - 6 = 0.$
14. $x^3 - 2x^2 - 5x + 6 = 0.$
15. $4x^3 - 33x - 20 = 0.$
16. $x^3 - 3x - 1 = 0.$
17. $x^3 - 5x^2 + 2 = 0.$
18. $x^3 - 6x^2 + 3x + 5 = 0.$
19. $x^3 + 3x^2 - 9x - 5 = 0.$
20. $x^3 - 3x^2 - 2x + 5 = 0.$

80. SOLUTION OF THE QUARTIC EQUATION

To solve the general quartic equation

$$ax^4 + bx^3 + cx^2 + dx + e = 0, \tag{62}$$

we set $x = y - \frac{b}{4a}$, a transformation that increases the roots by $\frac{b}{4a}$ and gives us a quartic equation of the form

$$y^4 + py^2 + qy + r = 0, \tag{63}$$

where the term in y^3 is lacking. Let us express the left member of this equation as the product of two quadratic factors as follows:

$$\begin{aligned} y^4 + py^2 + qy + r &= (y^2 + 2hy + m)(y^2 - 2hy + n) \\ &= y^4 + (m + n - 4h^2)y^2 + 2h(n - m)y + mn. \end{aligned} \tag{64}$$

It is possible to solve for m, n, and h from the three equations formed by equating the coefficients of like powers of y. The three equations are

$$\begin{aligned} m + n - 4h^2 &= p, & n + m &= p + 4h^2, \\ 2h(n - m) &= q, \quad \text{or} & n - m &= \frac{q}{2h}, \\ mn &= r, & mn &= r. \end{aligned} \tag{65}$$

Solving the first two equations for m and n in terms of p, q, and h, we have

$$m = \frac{1}{2}\left[p + 4h^2 - \frac{q}{2h}\right], \tag{66}$$

$$n = \frac{1}{2}\left[p + 4h^2 + \frac{q}{2h}\right]. \tag{67}$$

Since, by Equation (65)

$$mn = r,$$

then

$$\frac{1}{4}\left[p + 4h^2 - \frac{q}{2h}\right]\left[p + 4h^2 + \frac{q}{2h}\right] = r;$$

simplifying, gives

$$64h^6 + 32ph^4 + 4(p^2 - 4r)h^2 - q^2 = 0. \tag{68}$$

This is a cubic equation in h^2 and can be solved for h^2 by the method given in Article 78 or Article 79. Having the value of h, we can find m and n by substituting this value of h in Equations (66) and (67). The four roots of Equation (63) can be obtained by solving the two quadratic equations

$$y^2 + 2by + m = 0, \quad \text{and} \quad y^2 - 2hy + n = 0. \tag{69}$$

The roots of the general quartic Equation (62) are found by adding $-\frac{b}{4a}$ to each of the roots of Equation (69).

Example. Solve $x^4 - 4x^3 + 5x^2 - 4x + 4 = 0$.

Solution. Set $x = y - \frac{b}{4a} = y + 1$. This is equivalent to finding an equation whose roots are 1 unit less than the roots of the given equation, and by Article 76 this equation can be found as follows:

$$\begin{array}{r|l}
1 - 4 + 5 - 4 + 4 & \underline{1} \\
\quad + 1 - 3 + 2 - 2 & \\
\hline
1 - 3 + 2 - 2 + 2 & \\
\quad + 1 - 2 + 0 & \\
\hline
1 - 2 + 0 - 2 & \\
\quad + 1 - 1 & \\
\hline
1 - 1 - 1 & \\
\quad + 1 & \\
\hline
1 + 0 &
\end{array}$$

The transformed equation is $y^4 - y^2 - 2y + 2 = 0$. Thus, $p = -1$, $q = -2$, $r = 2$. Substituting these values in Equation (68), we have

$$64h^6 - 32h^4 - 28h^2 - 4 = 0,$$

or

$$16h^6 - 8h^4 - 7h^2 - 1 = 0. \tag{70}$$

By observation it is evident that $h^2 = 1$. Hence,

$$m = \tfrac{1}{2}[-1 + 4 + 1] = 2,$$

and

$$n = \tfrac{1}{2}[-1 + 4 - 1] = 1.$$

Therefore, by Equation (64)

$$y^4 - y^2 - 2y + 2 = (y^2 + 2hy + m)(y^2 - 2hy + n)$$
$$= (y^2 + 2y + 2)(y^2 - 2y + 1) = 0.$$

Hence,

$$y^2 - 2y + 1 = 0, \quad \text{and} \quad y^2 + 2y + 2 = 0.$$

Solving for y, we have $y = 1, 1$ and $-1 \pm i$.

Since $x = y + 1$, the four roots of the given equation are $x = 2, 2$, and $\pm i$.

EXERCISES

Solve each of the following equations.

1. $x^4 + 4x^3 + 7x^2 + 8x + 10 = 0.$
2. $x^4 - 4x^3 + 7x^2 - 4x + 6 = 0.$
3. $x^4 + 8x^3 + 17x^2 + 8x + 16 = 0.$
4. $x^4 + 4x^3 - x^2 - 6x + 18 = 0.$
5. $x^4 + 4x^3 - 3x^2 - 10x + 8 = 0.$
6. $x^4 + 8x^3 + 13x^2 - 8x - 14 = 0.$
7. $x^4 - 4x^3 - 5x^2 + 22x - 8 = 0.$
8. $16x^4 - 32x^3 + 56x^2 - 24x + 33 = 0.$
9. $x^4 - 4x^3 + 8x^2 - 7x + 4 = 0.$
10. $x^4 + 4x^3 + 4x^2 - 3x - 6 = 0.$
11. $16x^4 + 32x^3 - 8x^2 + 24x - 15 = 0.$
12. $x^4 - 12x^3 + 43x^2 - 38x - 24 = 0.$
13. $x^4 + 4x^3 + 2x^2 + x - 2 = 0.$
14. $x^4 - 8x^3 + 20x^2 - 11x - 14 = 0.$
15. $x^4 - 4x^3 + 4x^2 - 1 = 0.$
16. $x^4 - 2x^3 - 2x^2 + 2x + 1 = 0.$
17. $x^4 - 6x^3 + 14x^2 - 14x + 5 = 0.$
18. $x^4 - 4x^3 + 5x^2 - 2x - 2 = 0.$
19. $x^4 - 2x^3 + 7x^2 - 8x + 12 = 0.$
20. $x^4 + 4x^3 + 9x^2 + 10x + 6 = 0.$

CHAPTER XIII

Systems of Equations Involving Quadratics

CASE I. *One linear equation and one quadratic equation.*
CASE II. *Both equations of the form* $ax^2 + by^2 = c$.
CASE III. *Both equations of the form* $ax^2 + bxy + cy^2 = d$.

81. GRAPHING QUADRATIC EQUATIONS

The type of functions studied thus far were **single-valued**; that is, for every value of the independent variable x there was only one value for the dependent variable y. We shall now study **double-valued functions**; that is, functions in which one of the variables has two values for some or all of the values of the other variable. To illustrate, if we solve the equation $x^2 + y^2 = 25$ for y in terms of x, then y is a double-valued function of x because

$$y = \pm\sqrt{25 - x^2}.$$

Thus, we see that for each value of x between -5 and $+5$ there are two real and distinct values for y; the one value of y is positive, the other value of y is negative, and the numerical value of the two are equal. Thus, the graph of $x^2 + y^2 = 25$ is symmetric about the x-axis. From Fig. 35 it is easy to see that the graph of

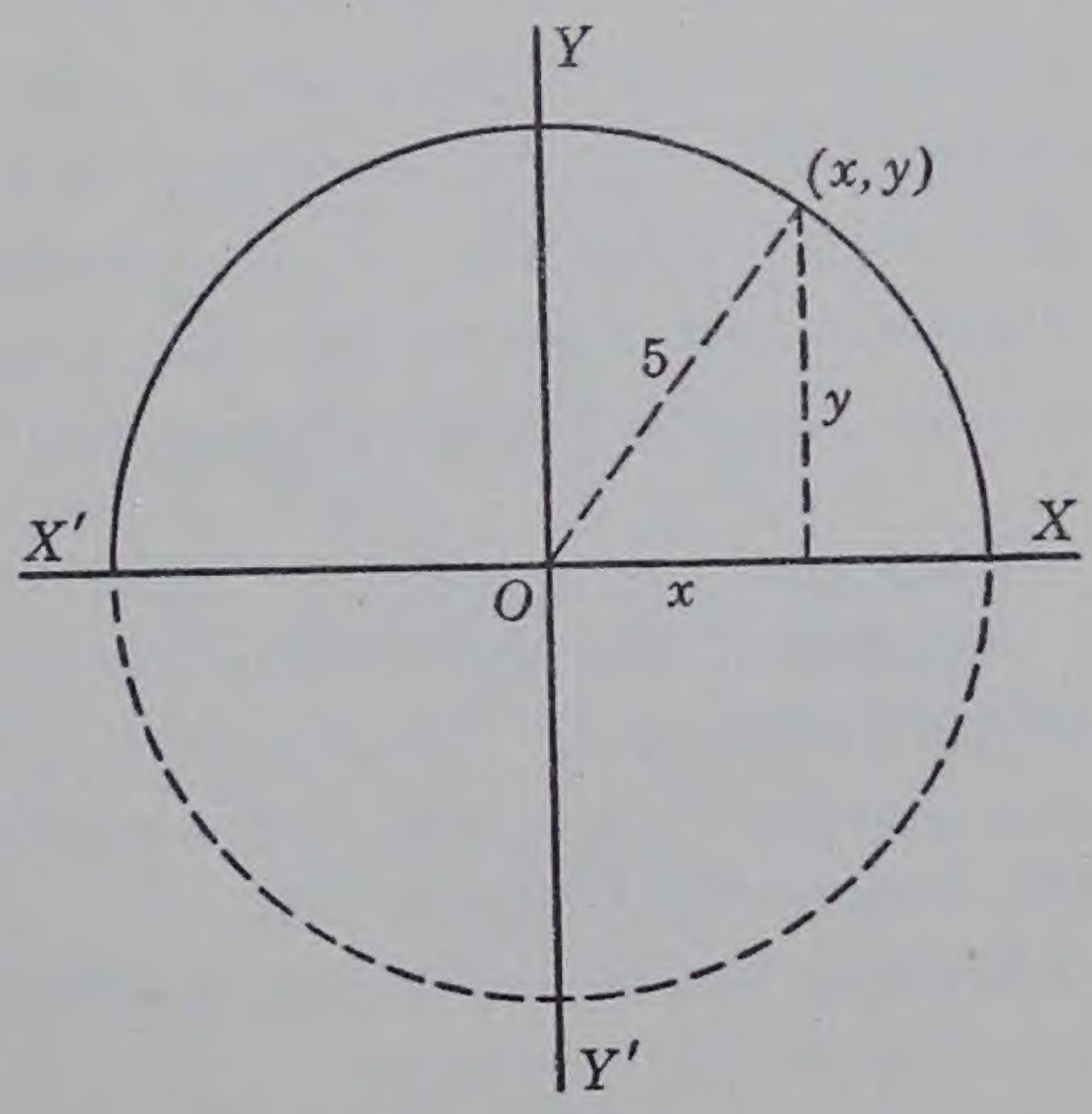


FIG. 35.

$$x^2 + y^2 = 25$$

is a circle, since it is the locus of a point 5 units from the origin.

The graph of

$$y = \sqrt{25 - x^2}$$

is a semicircle above the x-axis, and the graph of

$$y = -\sqrt{25 - x^2}$$

is a semicircle below the x-axis, as shown in Fig. 35.

If we solve the equation $x^2 + y^2 = 25$ for x, in terms of y, then x is a double-valued function of y because

$$x = \pm\sqrt{25 - y^2}.$$

There are two real and distinct values of x for each value of y between -5 and $+5$, and we see that the graph is also symmetric about the y-axis.

The most general form of a quadratic equation in two variables, x and y, is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

where at least one of the three coefficients A, B, and C, is not zero. Equations of this type can be graphed by calculating a table of points. After the points are plotted, a smooth graph can be drawn through the plotted points. This table of points can be calculated either by assigning values to the one variable and then solving for the other variable, or by solving the given equation for the one variable in terms of the other variable, and then assigning values to the variable appearing in the solution. To illustrate:

Example. Graph $2x^2 + 7xy + y^2 - 5x - 10y + 5 = 0$.

Solution. We shall solve this equation for y in terms of x and then assign values to x. To do this, it is necessary to write the given equation as a quadratic in y, and then we can use the quadratic formula, Equation (9), Article 60, Chapter XI.

Writing the equation as a quadratic in y, we have

$$y^2 + (7x - 10)y + (2x^2 - 5x + 5) = 0;$$

applying the formula gives

$$y = \frac{-(7x - 10) \pm \sqrt{(7x - 10)^2 - 4(2x^2 - 5x + 5)}}{2};$$

simplifying gives

$$y = \frac{10 - 7x \pm \sqrt{41x^2 - 120x + 80}}{2}.$$

Then

$$\text{if } x = 0, \quad y = \frac{10 \pm \sqrt{80}}{2} = 9.47 \text{ or } 0.53;$$

$$\text{if } x = 1, \quad y = \frac{3 \pm 1}{2} = 2 \text{ or } 1;$$

$$\text{if } x = 2, \quad y = \frac{-4 \pm 2}{2} = -1 \text{ or } -3;$$

$$\text{if } x = 3, \quad y = \frac{-11 \pm \sqrt{89}}{2} = -0.8 \text{ or } -10.2;$$

$$\text{if } x = 4, \quad y = \frac{-18 \pm 16}{2} = -1 \text{ or } -17;$$

$$\text{if } x = -1, \quad y = \frac{17 \pm \sqrt{241}}{2} = 16.3 \text{ or } 0.7;$$

$$\text{if } x = -2, \quad y = \frac{24 \pm 22}{2} = 23 \text{ or } 1;$$

$$\text{if } x = -3, \quad y = \frac{31 \pm \sqrt{809}}{2} = 29.7 \text{ or } 1.3.$$

These points are plotted and the graph is drawn in Fig. 36.

As an aid in drawing an approximate graph of a quadratic equation in two variables, we shall give without proof the following properties of such graphs. These properties can be found in any analytic geometry textbook.

The graph, if it is a real locus, of any quadratic equation in two variables is called a conic and has the form of a cross-section of a right circular cone intersected by a plane. (This includes the limiting cases of a cone.)

FIG. 36.

The conics are classified as follows:

1. *Two straight lines* (intersecting, parallel, or coinciding).
2. *Circle.*
3. *Ellipse* (including a point).
4. *Parabola.*
5. *Hyperbola.*

The form of the graph of each of the different conics is given in Fig. 37.

Furthermore, if the quadratic equations have the special forms indicated in Fig. 37, then for a, b, and c positive, the graphs of such equations will have the indicated relation with respect to

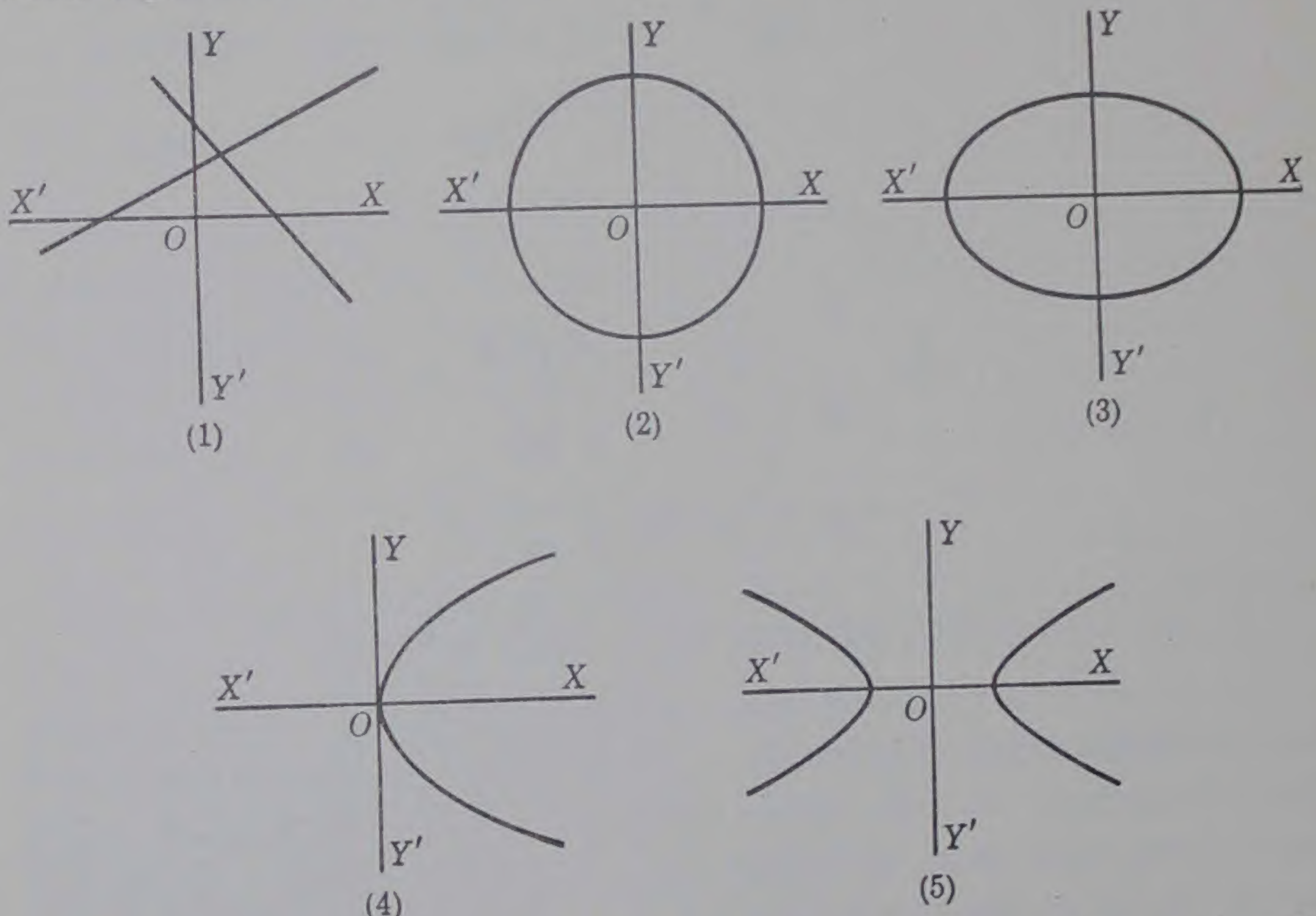


FIG. 37. (1) Two straight lines. $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$.
(2) Circle. $x^2 + y^2 = r^2$. (3) Ellipse. $ax^2 + by^2 = c$. (4) Parabola. $y^2 = 4ax$.
(5) Hyperbola. $ax^2 - by^2 = c$.

the coördinate axes. We may call these equations the standard forms for the conics.

1. The graph of $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$ is the graph of the two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. The two lines are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, otherwise the two lines intersect.

2. The radius of the circle $x^2 + y^2 = r^2$ is r.

3. The graph of the ellipse $ax^2 + by^2 = c$ is oblong about the x-axis if $a < b$, but is oblong about the y-axis if $a > b$.

4. The graph of the parabola $y^2 = 4ax$ is symmetric about the positive x-axis, and the graph of $y^2 = -4ax$ is symmetric about the negative x-axis.

The graph of the parabola $x^2 = 4ay$ is symmetric about the

positive y-axis, and the graph of $x^2 = -4ay$ is symmetric about the negative y-axis.

5. The graph of the hyperbola $ax^2 - by^2 = c$ is symmetric about the x-axis, and the graph of $ay^2 - bx^2 = c$ is symmetric about the y-axis.

An approximate graph of an equation in the standard form can be drawn quite readily by using the foregoing ideas and by determining the x- and y-intercepts, if they exist, along with a few other points.

It may happen that the locus of a quadratic equation in two variables is imaginary; that is, there are equations that are not satisfied by any real values of the two variables. In such cases the locus cannot be represented by a graph in the real plane.

The graph of the general quadratic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, if it is a real locus, will be one of the forms given in Fig. 37, but will be located differently with respect to the coördinate axes.

If $B^2 - 4AC < 0$, the graph is an *ellipse* or a *circle* (including a point).

If $B^2 - 4AC = 0$, the graph is a *parabola* or *two parallel lines.*

If $B^2 - 4AC > 0$, the graph is a *hyperbola* or *two intersecting straight lines.*

82. SOLUTION OF SYSTEMS OF EQUATIONS INVOLVING QUADRATICS

Problems involving one linear and one quadratic equation were solved by the Egyptians, and the Greeks studied the geometry of such problems centuries before Christ. But it was Diophantus, in the third century, who was the first to consider seriously an algebraic treatment of two quadratic equations, and these equations were of special types.

By a solution of a system of two quadratic equations in two unknowns is meant finding a pair of values of the unknowns that will satisfy each equation of the system. Thus, *in any complete solution the unknowns must be grouped in correct pairs.*

If the pair of numbers in a solution are real, they represent the coördinates of a point of intersection of the two graphs. If one or both parts of a solution are imaginary or infinite, then the solution does not represent a point on either graph.

To solve a system of two quadratic equations in two unknowns is to find all the pairs of values, whether real or imaginary, that will satisfy each of the two given equations.

It can be shown that *if the number of solutions in any system of two equations in two unknowns is finite, then the number of solutions is equal to the product of the degrees of the two equations.*

If the two given equations of a system have a common polynomial factor, then the system will have an infinite number of solutions. If the two equations of a system are inconsistent, there will be no solutions. If some of the solutions are infinite, the number of solutions of a given system may appear to be fewer than the product of the degrees of the two equations, since infinite solutions usually are not recorded.

The graphs of the two given equations of a system of two quadratic equations in two unknowns have one of the following properties:

1. If a complete solution consists of four distinct pairs of real numbers, then the graphs of the two equations intersect in four real points.

2. If a complete solution contains two like pairs of real values, then the graphs of the two equations are tangent at this point.

3. If each pair of values in a complete solution contains an imaginary number, then the two graphs will not intersect at any real point.

4. If two pairs of values in the solution are real numbers and the other two pairs of values contain imaginary numbers, then the two graphs intersect in only two real points.

Let us consider the following system composed of the two general equations

$$f(x, y) = 0, \tag{2}$$

and

$$F(x, y) = 0, \tag{3}$$

in the two unknowns x and y.

If we multiply each member of the first equation by k and each member of the second equation by K and then add the members of the two resulting equations, we have

$$\begin{array}{r} kf(x, y) = 0 \\ KF(x, y) = 0 \\ \hline kf(x, y) + KF(x, y) = 0. \end{array} \tag{4}$$

If (x_0, y_0) is a solution of the given system of Equations (2) and (3), then (x_0, y_0) must satisfy each of the given equations and hence $f(x_0, y_0) = 0$ and $F(x_0, y_0) = 0$. Therefore,

$$kf(x_0,y_0) + KF(x_0,y_0) = 0.$$

Geometrically, this means that the graph of the resulting equation $kf(x, y) + KF(x, y) = 0$ passes through all the points of intersection of the graphs of the two given equations, $f(x, y) = 0$ and $F(x, y) = 0$. From this we see that it is possible to obtain the solutions of a given system of two equations by solving the given system of equations, or by solving a system composed of a derived equation such as Equation (4) and either of the two given equations, or by solving a system composed of two derived equations.

The process for solving a system of two general quadratic equations may be rather long and difficult. However, there are certain special systems of equations that are easily solved. The most common of these systems are the following:

CASE I. *A system of one linear equation and one quadratic equation.*

CASE II. *A system in which both equations are of the form* $ax^2 + by^2 = c$.

CASE III. *A system in which both equations are of the form* $ax^2 + bxy + cy^2 = d$.

These coefficients may be positive or negative constants, or zero.

Case I: One Linear Equation and One Quadratic Equation

A system of one linear and one quadratic equation can always be solved by solving the linear equation for one of the unknowns in terms of the other unknown and substituting the result in the quadratic equation. The resulting equation is a quadratic in one unknown which can be solved for this unknown; then by substituting these values in the linear equation we have the corresponding values for the other unknown.

The complete solution of a system of one linear and one quadratic equation in two unknowns consists of two pairs of values. (The product of the degrees of the two equations is $1 \times 2 = 2$.) Furthermore, since the graph of the linear equation is a straight line, the two graphs have the following properties:

1. *The straight line intersects the graph of the quadratic equation in two distinct points if the two solutions are real and distinct.*

2. *The straight line is tangent to the graph of the quadratic equation if both pairs in the solution are real and equal.*

3. *The straight line neither intersects nor touches the graph of the quadratic equation if each of the two solutions contains imaginary numbers.*

Example 1. Solve the system of equations

$$x^2 + y^2 - 4x + 6y - 12 = 0, \tag{I}$$

$$x - y - 4 = 0. \tag{II}$$

Algebraic Solution. Solving Equation (II) for y, we obtain

$$y = x - 4;$$

substituting in (I) gives

$$x^2 + (x - 4)^2 - 4x + 6(x - 4) - 12 = 0;$$

simplifying gives

$$x^2 - 3x - 10 = 0;$$

factoring, we have

$$(x - 5)(x + 2) = 0.$$

Hence,

$$x = 5 \quad \text{or} \quad x = -2,$$

and since $y = x - 4$,

$$y = 1 \quad \text{or} \quad y = -6,$$

or the solutions may be written in the form (5, 1) and (−2, −6).

If we substitute $x = 5$ in the quadratic equation (I) instead of substituting in the linear equation (II), then

$$25 + y^2 - 20 + 6y - 12 = 0;$$

simplifying gives

$$y^2 + 6y - 7 = 0;$$

factoring, we have

$$(y - 1)(y + 7) = 0;$$

solving gives,

$$y = 1 \quad \text{or} \quad y = -7.$$

Hence the solutions appear to be (5, 1) and (5, −7). The points (5, 1) and (5, −7) are on the circle and their coördinates satisfy Equation (I). However, the point (5, −7) does not lie on the straight line, and hence its coördinates do not satisfy Equation (II). Therefore, (5, −7) is not a solution. In a similar manner, if we substitute $x = -2$ in Equation (I), then from the resulting equation we obtain the solutions (−2, −6) and

(−2, 0). However, the coördinates of the point (−2, 0) satisfy Equation (I), but they do not satisfy the linear equation (II). Hence (−2, 0) is not a solution. This is illustrated by the graphs in Fig. 38.

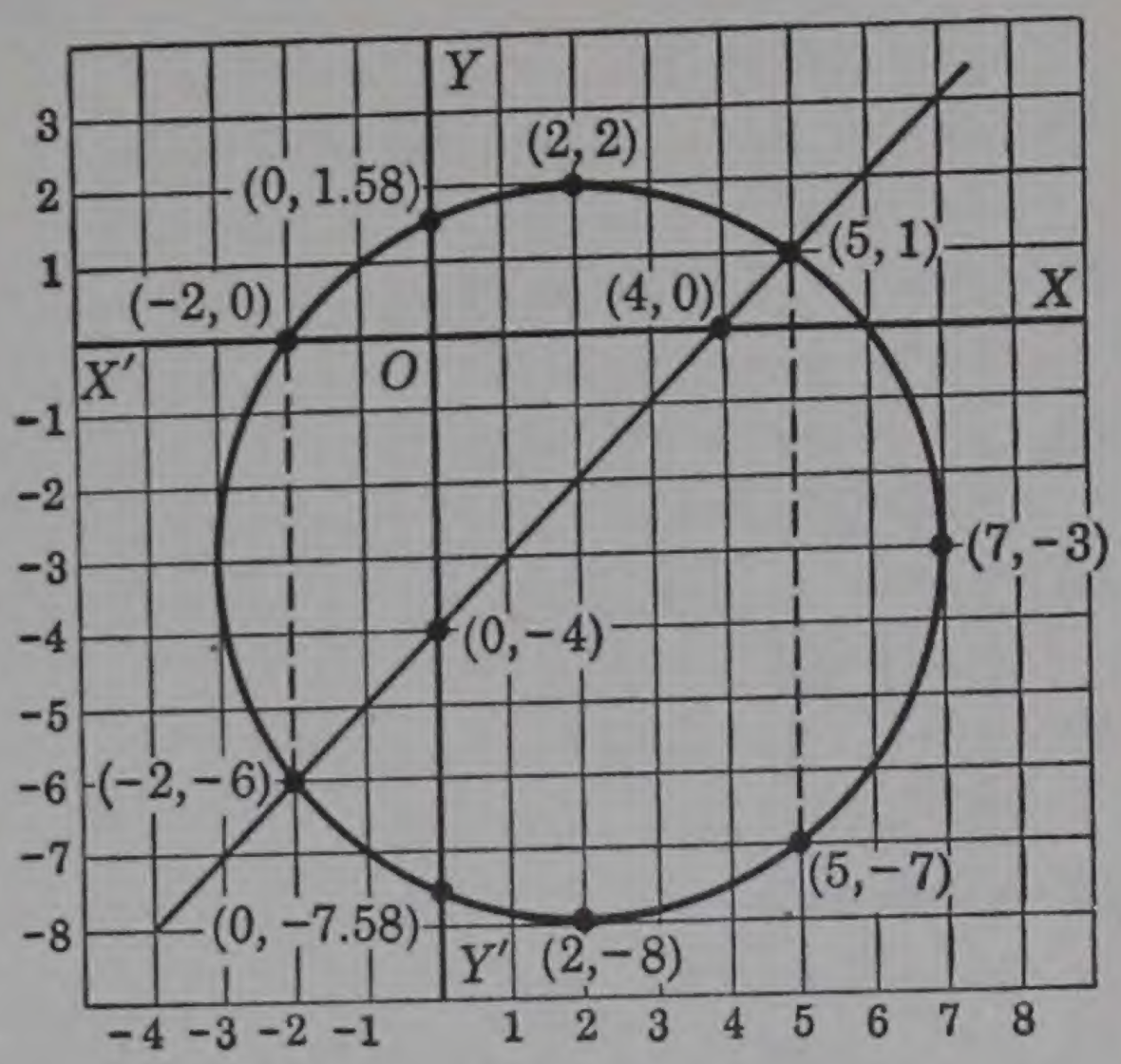


FIG. 38.

Thus, *after finding the values of the one unknown in any system of one linear and one quadratic equation, these values should always be substituted in the linear equation and not in the quadratic equation to find the value of the other unknown.*

Graphical Solution. The graph of $x^2 + y^2 - 4x + 6y - 12 = 0$ can be obtained by plotting points whose coördinates are found in the following manner. Let us solve this equation for y in terms of x and then assign values to x. Solving for y we obtain

$$y = -3 \pm \sqrt{21 + 4x - x^2}.$$

Hence

when $x = -2$,	$y = -3 \pm \sqrt{9} = 0$	or	$y = -6$;
when $x = 0$,	$y = -3 \pm \sqrt{21} = 1.58$	or	$y = -7.58$;
when $x = 2$,	$y = -3 \pm \sqrt{25} = 2$	or	$y = -8$;
when $x = 5$,	$y = -3 \pm \sqrt{16} = 1$	or	$y = -7$;
when $x = 7$,	$y = -3 \pm \sqrt{0} = -3$.		

The graph of $x - y - 4 = 0$ is a straight line with x-intercept 4 and y-intercept 4. The two curves intersect at the points (5, 1) and (−2, −6). These two pairs of values are the solutions of the given system of equations. (See Fig. 38.)

EXERCISES

Solve each of the following systems graphically and algebraically.

1. $x^2 + y^2 = 25$,
$x - y + 1 = 0$.

2. $9x^2 + 25y^2 = 225$,
$3x - 5y + 3 = 0$.

3. $2x^2 - y^2 = 14$,
$x + 2y = 7$.

4. $y^2 = 8x$,
$y + 2x - 8 = 0$.

5. $xy = 10$,
$y - x + 3 = 0$.

6. $4x^2 + 25y^2 = 100$,
$2x - 5y + 10 = 0$.

7. $x^2 + y^2 = 16$,
$x + y = 6$.

8. $9x^2 - 16y^2 = 144$,
$3x - 4y - 4 = 0$.

9. $x^2 + 3xy + 4y^2 = 64,$
$x - y = 4.$

10. $y^2 = 24x,$
$4y = 9x - 6.$

11. $y^2 = 8x,$
$y = 4x - 4.$

12. $x^3 + y^3 = 19,$
$x + y = 1.$

13. $4x^2 - 3y^2 - 5x - 2y + 7 = 0,$
$2x + y + 1 = 0.$

14. $16x^2 - 7y^2 = -48,$
$4x - y - 12 = 0.$

15. $xy - 3x + 2y = 6,$
$x + 2y = 1.$

16. $2x^2 - 3xy + 5x - 6y + 2 = 0,$
$2x + 3y = 11.$

17. $2x^2 - 5y^2 = 5,$
$2x - 3y + 1 = 0.$

18. $x^2 + 2xy + 3y^2 = 19,$
$x - 2y = 8.$

19. $x^2 + 8xy - y^2 - 4x - 14y + 19 = 0,$
$x + 3y - 1 = 0.$

20. $3x^2 + 12xy + y^2 - 8x - 18y + 13 = 0,$
$x - 3y + 5 = 0.$

Case II: Both Equations of the Form $ax^2 + by^2 = c$

If both equations of a system contain only the squares of the unknowns and constants, then it is possible to solve for the square of each unknown as in the case of a system of two linear equations. The four solutions consist of pairing the values of the two unknowns in all possible ways.

Since both equations are of the second degree, there will be $2 \times 2 = 4$ solutions.

Example 2. Solve the system of equations

$$9y^2 - 4x^2 = 108, \quad \text{(I)}$$
$$4y^2 + x^2 = 73. \quad \text{(II)}$$

Algebraic Solution. It is possible to solve for x^2 and y^2 in this system either by substitution, or by addition and subtraction, or by determinants. We shall solve for x^2 and y^2 by addition and subtraction; thus, by subtraction, we have

$$\begin{array}{rr} 36y^2 - 16x^2 = & 432 \\ 36y^2 + 9x^2 = & 657 \\ \hline -25x^2 = & -225 \end{array}$$

$$x^2 = 9.$$

Hence

$$x = \pm 3.$$

By addition, we have

$$\begin{aligned} 9y^2 - 4x^2 &= 108 \\ 16y^2 + 4x^2 &= 292 \\ \hline 25y^2 \qquad\quad &= 400 \end{aligned}$$

Hence

$$\begin{aligned} y^2 &= 16. \\ y &= \pm 4. \end{aligned}$$

To determine the solutions it is necessary to pair these values properly. From the discussion of the system of the two general equations (2) and (3), we see that the solutions of the system of equations in this example are obtained by finding the coördinates of the points of inter-

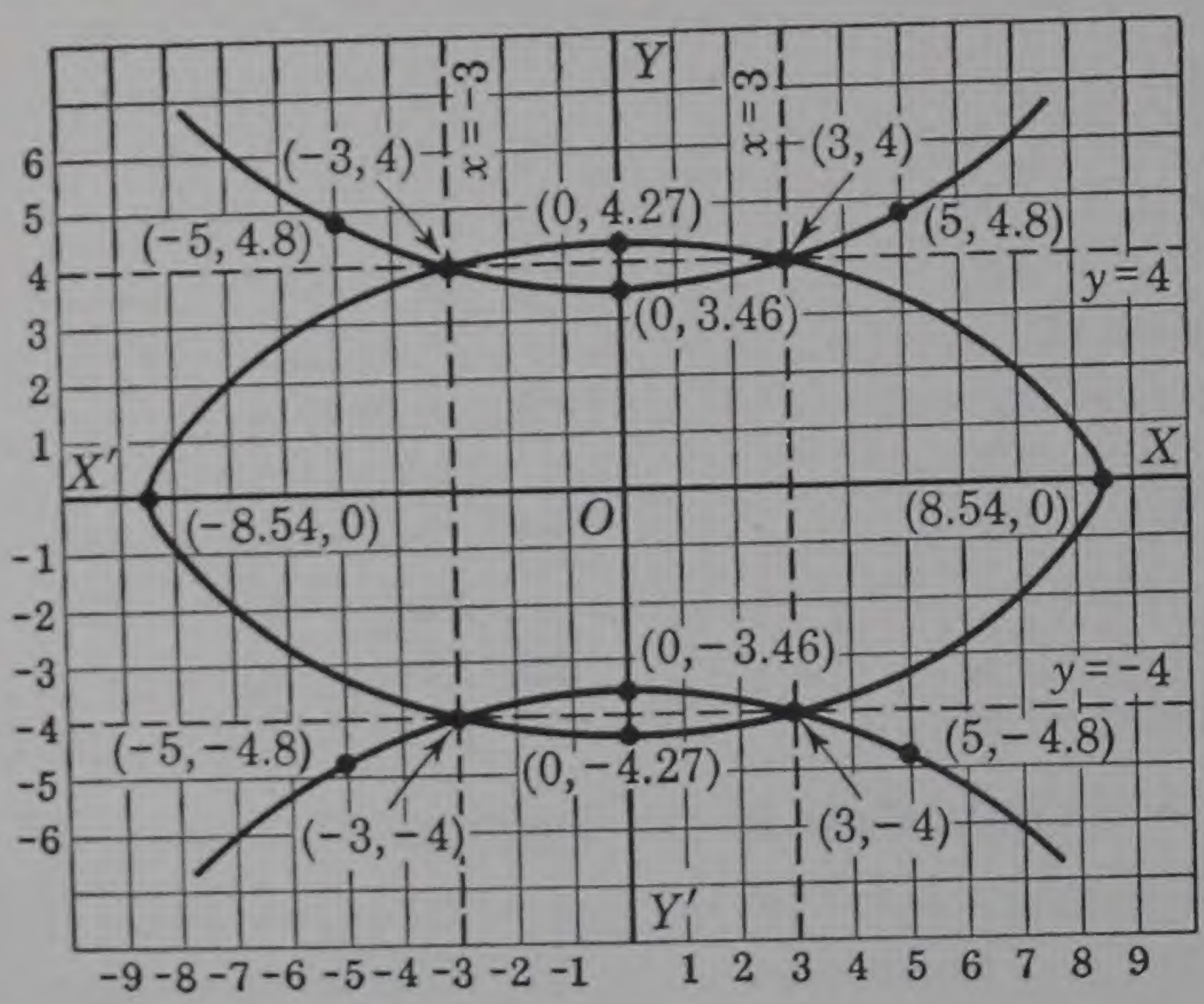


FIG. 39.

section of the two lines $x = 3$ and $x = -3$ with the two lines $y = 4$ and $y = -4$. (See Fig. 39.) Hence, the solutions are

$$\begin{aligned} x &= 3, \quad 3, -3, -3, \\ y &= 4, -4, \quad 4, -4. \end{aligned}$$

These solutions may be written in the form (3, 4), (3, −4), (−3, 4), and (−3, −4).

Graphical Solution. From Article 81 and Fig. 37, we see that the graph of $9y^2 - 4x^2 = 108$ is a hyperbola and the graph of $4y^2 + x^2 = 73$ is an ellipse. The y-intercepts of the hyperbola can be found by solving the equation $9y^2 = 108$ for y. Hence the y-intercepts are $y = \pm 3.46$. To find other points on this graph, let $x = \pm 5$; then $9y^2 = 108 + 100 = 208$, or $y = \pm 4.8$. Hence the graph of the hyperbola passes through the four points (5, 4.8), (5, −4.8), (−5, 4.8), and (−5, −4.8). These

four points and the y-intercepts are sufficient for us to draw an approximate graph of the hyperbola $9y^2 - 4x^2 = 108$. The x-intercepts of the ellipse $4y^2 + x^2 = 73$ are $x = \pm\sqrt{73} = \pm 8.54$. The y-intercepts are found by solving the equation $4y^2 = 73$ for y. Hence the y-intercepts are $y = \pm 4.27$. An approximate graph of the ellipse can be drawn through these four intercepts. The graphs of the two equations are given in Fig. 39, and the coördinates of their points of intersection are the four solutions. Hence the four solutions are $(3, 4)$, $(3, -4)$, $(-3, 4)$, and $(-3, -4)$.

EXERCISES

Solve each of the following systems graphically and algebraically.

1. $x^2 + y^2 = 25$,
$x^2 + 4y^2 = 52$.

2. $16y^2 + 5x^2 = 36$,
$25y^2 - 4x^2 = 9$.

3. $4x^2 + 16y^2 = 100$,
$16x^2 + 4y^2 = 100$.

4. $x^2 + y^2 = 25$,
$4x^2 + y^2 = 52$.

5. $4x^2 + 9y^2 = 36$,
$x^2 + y^2 = 49$.

6. $3x^2 - 5y^2 = 43$,
$x^2 + 2y^2 = 18$.

7. $x^2 + 3y^2 = 12$,
$4x^2 + 3y^2 = 12$.

8. $x^2 + y^2 = 8$,
$9x^2 + 16y^2 = 100$.

9. $5x^2 + 2y^2 = 20$,
$5y^2 - 8x^2 = 9$.

10. $25x^2 + 9y^2 = 225$,
$25x^2 - 9y^2 = 225$.

11. $4x^2 + 9y^2 = 72$,
$25y^2 - 4x^2 = 64$.

12. $16x^2 - 9y^2 = 55$,
$63y^2 - 32x^2 = -65$.

Case III: Both Equations of the Form $ax^2 + bxy + cy^2 = d$

If both equations contain only constants and terms of the second degree in the two unknowns, it is possible to solve these systems of equations by eliminating the constant terms. After factoring the resulting equation into two linear factors, form two linear equations by setting each factor equal to zero, and then solve the two systems that can be formed by using each one of the two linear equations and one of the two given equations.

Example 3. Solve the system of equations

$$\begin{aligned} &3x^2 + xy + y^2 = 15, &\text{(I)}\\ &3x^2 - 31xy + 5y^2 = -45. &\text{(II)}\end{aligned}$$

Solution. The constant terms can be eliminated by multiplying each term of Equation (I) by 3, and then adding the members of the resulting equation to the members of Equation (II). This gives

$$\begin{array}{r} 9x^2 + 3xy + 3y^2 = 45 \\ 3x^2 - 31xy + 5y^2 = -45 \\ \hline 12x^2 - 28xy + 8y^2 = 0 \end{array}$$

Factoring, we have

$$4(3x - y)(x - 2y) = 0.$$

Therefore,

$$3x - y = 0 \quad \text{or} \quad x - 2y = 0.$$

Using the first of these results we can form the following system of equations:

$$3x^2 + xy + y^2 = 15,$$
$$3x - y = 0.$$

Solving for y, gives

$$y = 3x;$$

substituting, we have

$$3x^2 + x(3x) + (3x)^2 = 15;$$

simplifying, gives

$$15x^2 = 15.$$

Hence,

$$x = 1 \quad \text{or} \quad x = -1;$$

and, since $y = 3x$,

$$y = 3 \quad \text{or} \quad y = -3.$$

Then, using the second linear factor, we can form another system of equations, as follows:

$$3x^2 + xy + y^2 = 15,$$
$$x - 2y = 0.$$

Solving for x gives

$$x = 2y,$$

substituting, we have

$$3(2y)^2 + (2y)y + y^2 = 15;$$

simplifying, gives

$$15y^2 = 15.$$

Hence,

$$y = 1 \quad \text{or} \quad y = -1;$$

and, since $x = 2y$,

$$x = 2 \quad \text{or} \quad x = -2.$$

Hence, the solutions are $(1, 3)$, $(-1, -3)$, $(2, 1)$, and $(-2, -1)$. The graphic solution is given in Fig. 40.

If one of the equations of a given system is of the form $ax^2 + bxy + cy^2 = 0$, then this equation can be factored into two linear factors from which the two linear equations can be formed without first eliminating the constant terms. It may be necessary to use the quadratic formula to solve the equation $ax^2 + bxy + cy^2 = 0$ for one of the unknowns. To illustrate, we shall solve this equation for x.

$$x = \frac{-by \pm \sqrt{b^2y^2 - 4acy^2}}{2a},$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\, y.$$

Any system of two quadratic equations in two unknowns can be solved in the following manner:

1. *If one of the two given equations is linear in one of the unknowns, then solve this equation for this unknown in terms of the other unknown and substitute the result in the other equation. The resulting equation contains but one unknown and can be solved for this unknown.*

2. *If neither of the two given equations is linear in one of the unknowns, then eliminate the square of one unknown. If the resulting equation is linear in one of the two unknowns, solve the equation for this unknown in terms of the other unknown, then substitute the result in either of the two given equations, and solve the resulting equation. If the derived equation contains but one unknown, it can be solved for this unknown and the corresponding values for the other unknown can be found by substituting these values in either of the two given equations.*

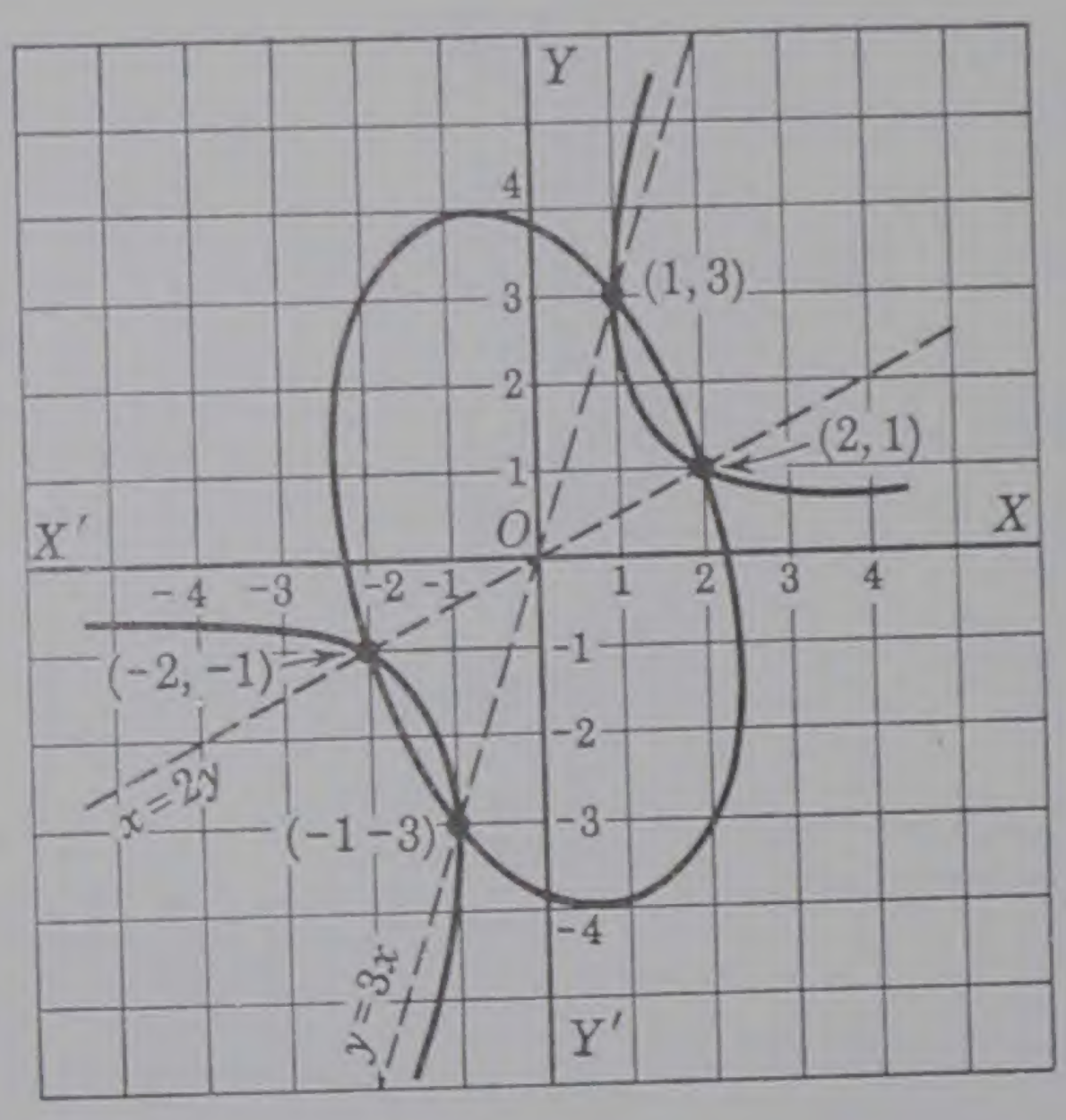


FIG. 40.

After clearing the resulting equation of fractions and radicals, if any, the final equation may be a fourth degree equation* in one unknown whose real roots can be found by methods described in Chapter XII. Sometimes the final equation can be solved as a quadratic equation. The corresponding values for the other unknown can be found by substituting these roots in the derived equation.

* If this final equation is a quadratic equation, that is, if the coefficients of x^4 and x^3 are zero, then two roots of this final equation are infinite. Therefore, two of the solutions will be infinite, and in these cases we usually write only the two remaining solutions.

We shall illustrate the general method for solving a system of two quadratic equations in two unknowns by solving the following system of equations.

Example 4. Solve the system of equations

$$2x^2 + 7xy + y^2 - 5x - 10y + 5 = 0, \quad \text{(I)}$$
$$x^2 - xy + 2y^2 - x + 4y - 14 = 0. \quad \text{(II)}$$

Solution. Eliminating y^2 by multiplying the members of Equation (I) by 2 and then subtracting the members of Equation (II) from the result, we have

$$\begin{array}{rrrrrrrr} 4x^2 & + 14xy & + 2y^2 & - 10x & - 20y & + 10 & = 0 \\ x^2 & - xy & + 2y^2 & - x & + 4y & - 14 & = 0 \\ \hline 3x^2 & + 15xy & & - 9x & - 24y & + 24 & = 0 \end{array}$$

dividing by 3, gives

$$x^2 + 5xy - 3x - 8y + 8 = 0;$$

solving for y, gives

$$(8 - 5x)y = x^2 - 3x + 8;$$

hence

$$y = \frac{x^2 - 3x + 8}{8 - 5x}.$$

Substituting in (I), we have

$$2x^2 + 7x\left(\frac{x^2 - 3x + 8}{8 - 5x}\right) + \left(\frac{x^2 - 3x + 8}{8 - 5x}\right)^2 - 5x - 10\left(\frac{x^2 - 3x + 8}{8 - 5x}\right) + 5 = 0;$$

simplifying, gives

$$16(x^4 - 5x^3 + 20x - 16) = 0.$$

Solving this equation by using synthetic division, we have

$$\begin{array}{rrrrrr} 1 & - 5 & + 0 & + 20 & - 16 & \lfloor 1 \\ & + 1 & - 4 & - 4 & + 16 & \\ \hline 1 & - 4 & - 4 & + 16 & & \lfloor 2 \\ & + 2 & - 4 & - 16 & & \\ \hline 1 & - 2 & - 8 & & & \lfloor 4 \\ & + 4 & + 8 & & & \\ \hline 1 & + 2 & & & & \end{array}$$

Therefore,

$$x^4 - 5x^3 + 20x - 16 = (x - 1)(x - 2)(x - 4)(x + 2) = 0,$$

and the roots are

$$x = 1,\ 2,\ 4,\ -2.$$

Since $y = \dfrac{x^2 - 3x + 8}{8 - 5x}$, then

$$\text{when } x = 1, \qquad y = \frac{1 - 3 + 8}{8 - 5} = 2;$$

$$\text{when } x = 2, \qquad y = \frac{4 - 6 + 8}{8 - 10} = -3;$$

$$\text{when } x = 4, \qquad y = \frac{16 - 12 + 8}{8 - 20} = -1;$$

$$\text{when } x = -2, \qquad y = \frac{4 + 6 + 8}{8 + 10} = 1.$$

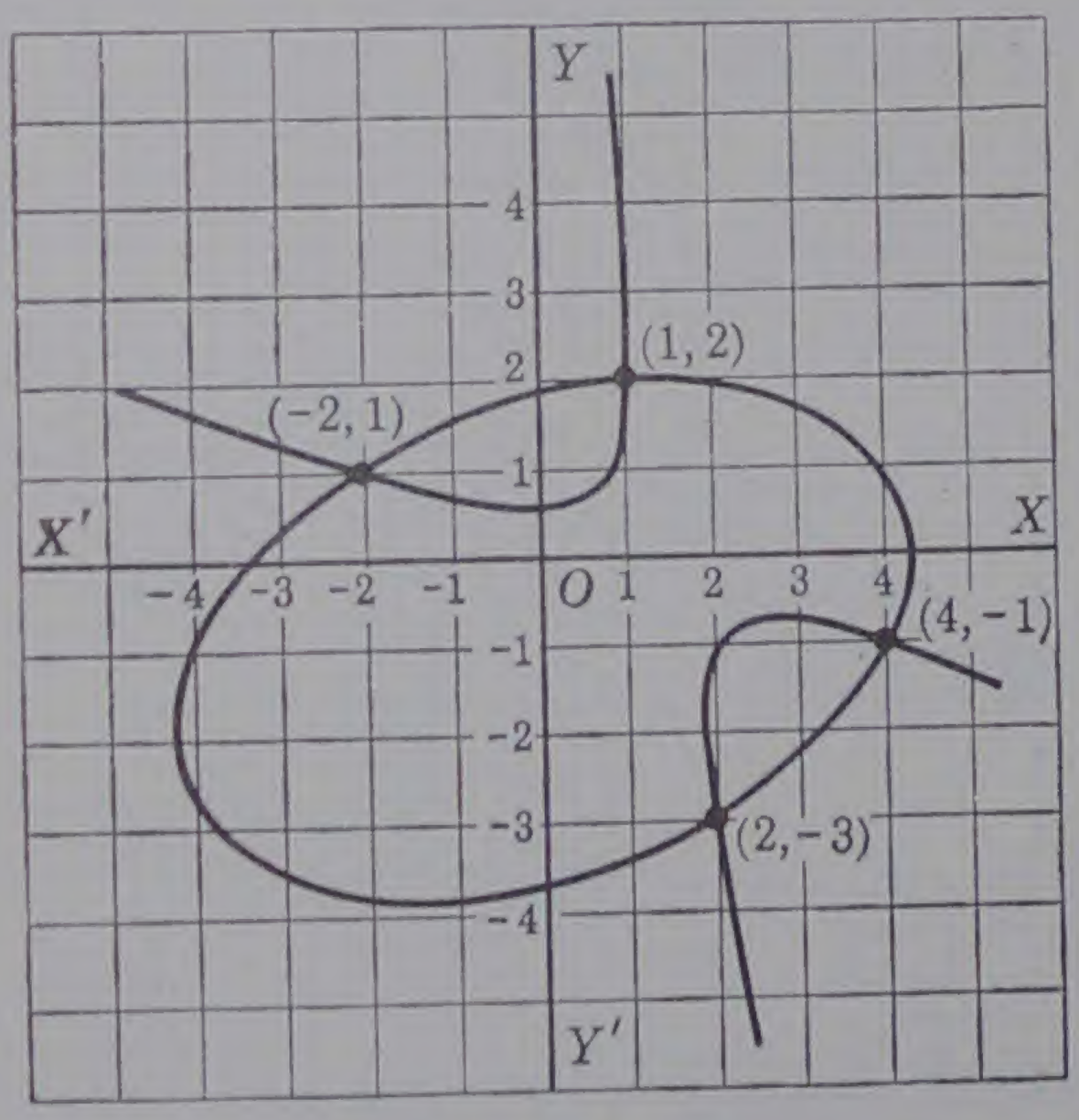


FIG. 41.

These solutions may be written as (1, 2), (2, −3), (4, −1), and (−2, 1).

The solution by graphs is given in Fig. 41. Equation (I) was graphed from data used in Fig. 36. As an exercise the student should compute the data necessary to graph Equation (II).

Example 5. Let us solve Example 3 by this general method. The equations are

$$\text{(I)} \quad 3x^2 + xy + y^2 = 15,$$
$$\text{(II)} \quad 3x^2 - 31xy + 5y^2 = -45.$$

Solution. Subtracting Equation (II) from Equation (I) to eliminate x^2, we have

$$32xy - 4y^2 = 60;$$

solving for x gives

$$x = \frac{y^2 + 15}{8y}.$$

Substituting in (I) gives

$$3\left(\frac{y^2 + 15}{8y}\right)^2 + \left(\frac{y^2 + 15}{8y}\right)y + y^2 = 15;$$

simplifying, we have

$$3(y^4 + 30y^2 + 225) + 8y^4 + 120y^2 + 64y^4 = 960y^2;$$

collecting, gives

$$75y^4 - 750y^2 + 675 = 0;$$

factoring, we find

$$75(y^2 - 1)(y^2 - 9) = 0.$$

Then

$$y^2 = 1 \quad \text{or} \quad y^2 = 9,$$

hence,

$$y = \pm 1 \quad \text{or} \quad y = \pm 3.$$

Since $x = \dfrac{y^2 + 15}{8y}$, then

$$\text{when } y = \quad 1, \quad x = \frac{1 + 15}{8} = \quad 2;$$

$$\text{when } y = -1, \quad x = \frac{1 + 15}{-8} = -2;$$

$$\text{when } y = \quad 3, \quad x = \frac{9 + 15}{24} = \quad 1;$$

$$\text{when } y = -3, \quad x = \frac{9 + 15}{-24} = -1.$$

The solutions may be written as (2, 1), (−2, −1), (1, 3), and (−1, −3). The graphic solution is given in Fig. 40.

QUESTIONS

1. The graphs of quadratic equations in two variables are of what forms?

2. What is meant by a solution of a system of two equations in two unknowns, and how many solutions does such a system have?

3. Give three common classes of systems of equations containing quadratics, and give a method for solving each.

EXERCISES

Solve each of the following systems of equations (Exercises 1 through 18).

1. $xy = 6,$
$3x^2 + 2y^2 = 30.$

2. $x^2 + y^2 = 41,$
$xy = 20.$

3. $9x^2 + 25y^2 = 225,$
$2xy = 9.$

4. $x^2 + y^2 = 26,$
$xy = 5.$

5. $5x^2 - 12xy + 7y^2 = 3,$
$3x^2 - 12xy + 11y^2 = -1.$

6. $x^2 - xy + y^2 = 12,$
$x^2 + xy + y^2 = 28.$

7. $8x^2 - 3xy + 11y^2 = 392,$
$x^2 + xy - 2y^2 = -56.$

8. $3x^2 - xy = 36,$
$2xy - 4y^2 = -12.$

9. $x^2 + 2xy = -3,$
$2xy + y^2 = -8.$

10. $2x^2 + 12xy - 7y^2 = -25,$
$6x^2 - 4xy + 9y^2 = 75.$

11. $x^2 - 3xy + 2y^2 = 0,$
$4x^2 - 5xy + 10y^2 = 36.$

12. $x^2 + xy = 15,$
$xy + 3y^2 = 2.$

13. $x^2 + xy = 10,$
$3xy + 4y^2 = -9.$

14. $x^2 + 2xy = 7,$
$x^2 + y^2 = 10.$

15. $4x^2 + 25y^2 = 100,$
$5y^2 = 6xy.$

16. $6x^2 - 7xy - 3y^2 = 0,$
$x^2 - 2xy - 2y^2 = -11.$

17. $x^2 - 2xy - 7y^2 = -7,$
$3x^2 - 12xy - 37y^2 = -49.$

18. $x^2 - 6xy = 40,$
$x^2 + 6xy + 3y^2 = -5.$

MISCELLANEOUS EXERCISES AND PROBLEMS

Solve each of the following systems of equations (Exercises 1 through 18).

1. $x^2 - 2xy = 0,$
$2x^2 - xy + 2y^2 = 18.$

2. $x(y - 5) = 0,$
$y(x + 2) = 5.$

3. $x + 2y - 3x^2 = -12.$
$x + y - 2x^2 = -7.$

4. $x(y + 2) = 9,$
$y(x + 2) = 5.$

5. $2(x^2 + y^2) = 5xy,$
$2(x + y) = -3xy.$

6. $x^2 - xy = a^2 + b^2,$
$xy - y^2 = 2ab.$

7. $x^2 - 2x + 2y^2 = 1,$
$3x^2 - 7x + 6y^2 + 4y = 6.$

8. $x^2 - 6xy + 9y^2 - 1 = 0,$
$2x^2 - 3xy - 27y^2 - 3x + 12y + 7 = 0.$

9. $x^2 + xy - 6y^2 = 0,$
$x^2 + y^2 - 5y = 30.$

10. $3x^2 - 3y^2 - 4x = 0,$
$2x^2 + 2y^2 + 4x = 11.$

11. $(x + 6y)(x - y) = 4xy,$
$x^2 - 3y^2 + x - 2y = 1.$

12. $x^2 + y^2 + 2x - 3y + 2 = 0,$
$x^2 + y^2 + 6x + 5y + 14 = 0.$

13. $(x - 1)(y + 2) = 6,$
$(x + 2)(y - 1) = 12.$

14. $(x - 1)(y + 2) = -4,$
$(x + 4)(y - 3) = -49.$

15. $x^2 + y^2 - 3x + 7y - 58 = 0,$
$3x^2 + 3y^2 - 5x + 9y - 122 = 0.$

16. $6xy - 2x + 3y = -24,$
$4xy + 3x - y = -35.$

17. $5x^2 - 6xy + 7y^2 + 2x + 6y - 70 = 0,$
$2x^2 + 12xy - 17y^2 + 8x + 24y + 35 = 0.$

18. $5x^2 + 4xy + 7y^2 - 8x + 2y - 37 = 0,$
$3x^2 + 12xy + y^2 - 8x - 18y + 13 = 0.$

19. Find the points of intersection of the circle $x^2 + y^2 + x + y - 26 = 0$ and the straight line $9x + 5y + 7 = 0$.

20. Find the points of intersection of the circle $2x^2 + 2y^2 - 4x + 11y - 121 = 0$ and the straight line $2x - 9y + 37 = 0$.

21. Find the points of intersection of the two circles $x^2 + y^2 + x - y - 18 = 0$ and $x^2 + y^2 + 8x + 4y - 17 = 0$.

22. Find the points of intersection of the two circles $x^2 + y^2 - 3x + 3y - 10 = 0$ and $2x^2 + 2y^2 + 8x + 41y + 113 = 0$.

23. Find two numbers whose difference is 5 and whose product is 176.

24. The difference of the squares of two numbers is 119, and the difference of the numbers is 7. Find the numbers.

25. If the length and breadth of a rectangle are each increased by 2 rods, the area will then be 1280 square rods; if they are each diminished by 2 rods, the area will be 960 square rods. Find the length and breadth of the original rectangle.

26. The area of a rectangle is 360 square feet. If the length is increased by 12 feet and the width by 5 feet, the area will be doubled. Find the dimensions of the rectangle.

27. What must be the dimensions of a rectangle if the area is 80 square feet and the perimeter is 36 feet?

28. The area of a rectangle is 1200 square feet and the length of the diagonal is 50 feet. Find the dimensions of the rectangle.

29. Two automobiles A and B travel a distance of 320 miles at different rates. A can start 1 hour after B starts, overtake B in four hours, and reach the destination 1 hour ahead of B. Find their rates.

30. Two towns are 150 miles apart. From each town and at the same time an automobile starts for the other town and each travels at a uniform speed. Two hours later they meet, but the one automobile reaches its destination 1 hour and 40 minutes before the other automobile reaches its destination. Find the rate of each automobile.

31. A grocer buys a lot of eggs for \$16.20. He found 15 broken eggs. He sold the remaining eggs at a profit of 5 cents per dozen and made \$2.60 on the transaction. How many eggs did he buy, and what did he pay per dozen?

32. A merchant bought two lots of dresses and paid \$200 for each lot. The first lot cost him \$2 more per dress than the second lot, but he got 5 dresses more in the second lot than in the first. How many dresses did he buy, and what was the price of each dress?

33. A club wishing to raise a sum of \$120 discovered that it would cost them \$2 more apiece than they had expected, since 3 of the members refused to contribute. How many members were in the club?

34. A farmer baled 9 tons of hay. If he had made each bale 10 pounds heavier, he would have had 20 bales less. How many bales did he have and what was the weight of each bale?

35. In a rectangular meadow of 8 acres a farmer makes 22 rounds with a mowing machine which cuts a swath 6 feet wide; there remains $2\frac{2}{5}$ acres standing. Find the dimensions of the field in rods (160 square rods = 1 acre, $16\frac{1}{2}$ feet = 1 rod).

36. The product of two numbers is 180 less than the square of the one number and 144 more than the square of the other number. Find the two numbers.

37. An automobile traveling on a road which is one half level and one half uphill must travel 6 miles per hour slower uphill than on the level and can make the trip in $36\frac{2}{3}$ minutes. The return trip can be made in 30 minutes, and the rate downhill is 9 miles more per hour than on the level. Find the rate on the level, and the distance traveled one way.

38. A and B together can do a piece of work in 3 days. It takes A $1\frac{1}{4}$ days less to do one half of the work than it takes B to do all the work. How long does it take A and B each alone to do the work?

39. In a right triangle with sides 9, 12, and 15, find the length of the perpendicular from the right angle to the hypotenuse.

40. The sum of the numerator and denominator of a fraction is 17. If 1 is subtracted from the numerator and 4 is added to the denominator, the value of the fraction is decreased by $\frac{1}{6}$. Find the fraction.

41. Two women, A and B, spent \$2 each for coffee. A paid 20 cents more for 3 pounds than B paid for 4 pounds but B received 3 pounds more than A received. What did each pay per pound?

CHAPTER XIV

Arithmetic and Geometric Progresssions

FORMULAS

1. $l = a + (n - 1)d.$
2. $s = \frac{n}{2}(a + l).$
3. $l = ar^{n-1}.$
4. $s = \frac{a(1 - r^n)}{1 - r}.$
7. $s = \frac{a}{1 - r}.$

83. ARITHMETIC PROGRESSIONS

The idea of series is very ancient, dating back as early as 2000 B.C. The Egyptians at that time studied certain series such as those series whose terms form an arithmetic or geometric progression. Series have since played a very important role in the development of mathematics.

DEFINITION. *A series is an indicated sum of a succession of terms, each term of which is derived from the preceding terms by a fixed law.*

In some series the number of terms is finite and they are called **finite series;** in other series the number of terms is unlimited and they are called **infinite series.**

Perhaps the simplest type of series are those whose terms are formed by adding a given number to the preceding term, as:

$$1 + 2 + 3 + 4 + 5 + \cdots,$$
$$2 + 4 + 6 + 8 + 10 + \cdots,$$
$$7 + 4 + 1 - 2 - 5 - \cdots.$$

The terms of each such series form an arithmetic progression; that is, *an* ***arithmetic progression*** *is a sequence of numbers, called terms, each of which after the first is obtained by adding to the pre-*

ceding term a fixed number called the **common difference.** For example, each of the following is an arithmetic progression:

$$1, 2, 3, 4, 5, \cdots,$$
$$2, 4, 6, 8, 10, \cdots,$$
$$7, 4, 1, -2, -5, \cdots.$$

In the first of the foregoing progressions, each term after the first term is formed by adding 1 to the preceding term; in the second progression, each term after the first term is formed by adding 2 to the preceding term; and in the third progression, each term after the first is formed by adding -3 to the preceding term.

If an arithmetic progression is given, *the common difference is found by subtracting any term from the following term.* In the first progression of the foregoing set, the common difference is 1; in the second progression, the common difference is 2; and in the third progression, the common difference is -3.

Let a = the first term,
d = the common difference,
n = the number of terms to be considered,
l = the last or nth term, and
s = the sum of the first n term.

If a is the first term of an arithmetic progression and d is the common difference, then the second term is $a + d$, the third term is $a + 2d$, the fourth term is $a + 3d$, and so on. Finally the nth or last term is

$$l = a + (n - 1)d. \tag{1}$$

By the sum of an arithmetic progression is meant the sum of the first n terms. The series representing the sum may be written in either of the following forms:

$$s = a + (a + d) + (a + 2d) + \cdots + (l - 2d) + (l - d) + l,$$
$$s = l + (l - d) + (l - 2d) + \cdots + (a + 2d) + (a + d) + a.$$

Adding these two series we have

$$2s = (a + l) + (a + l) + (a + l) + \cdots + (a + l) + (a + l) + (a + l);$$

hence,

$$2s = n(a + l),$$

and

$$s = \frac{n}{2}(a + l). \tag{2}$$

The two formulas given by Equations (1) and (2) are sufficient to solve completely most of the elementary problems arising in arithmetic progressions. Each of the equations (1) and (2) contains four letters, and the two equations together contain all five letters. Therefore, *if the values of three letters of an arithmetic progression are given, it is possible to solve for the values of the other two letters by substituting the values of the three given letters in equations* (1) *and* (2), *and then solving the resulting two equations as a system of two equations in two unknowns.*

Before attempting to solve any problem in arithmetic progression it is necessary to ascertain the values of three of the five letters. To illustrate:

Example 1. Find the twentieth term and the sum of the first twenty terms of the arithmetic progression 8, 5, 2, -1, ...

Solution. In this example $a = 8$, $d = -3$, and $n = 20$. We substitute these values in Equation (1) to find the twentieth term, as follows:

$$l = 8 + (20 - 1)(-3) = 8 - 57 = -49.$$

Substituting $n = 20$, $a = 8$, and $l = -49$ in Equation (2), we then find that the sum of the first twenty terms is

$$s = \tfrac{20}{2}(8 - 49) = -410.$$

Example 2. How many terms of the progression 3, 7, 11, 15, . . . are necessary to give the sum 300?

Solution. In this example we have given $a = 3$, $d = 4$, and $s = 300$. Substituting these values in Equations (1) and (2), we have

$$l = 3 + (n - 1)4,$$

$$300 = \frac{n}{2}(3 + l).$$

Simplifying these two equations, we have

$$\left.\begin{aligned} &\text{(I)} && l - 4n = -1, \\ &\text{(II)} && nl + 3n = 600. \end{aligned}\right\}$$

Solving (I) for l gives

$$l = 4n - 1.$$

Substituting in (II), we have

$$n(4n - 1) + 3n = 600;$$

simplifying, gives

$$4n^2 + 2n - 600 = 0;$$

factoring, we get

$$2(n - 12)(2n + 25) = 0;$$

hence,

$$n = 12 \quad \text{or} \quad n = -\tfrac{25}{2}.$$

Since the number of terms must be a positive integer, the number of terms must be 12.

It also is possible to solve Example 2 by first deriving a formula expressing n in terms of the three given letters a, d, and s. Then the value of n can be found by substituting the values of a, d, and s in this formula. This formula can be found by eliminating l and between Equations (1) and (2) and then solving the resulting equation for n.

The two given equations are

$$l = a + (n - 1)d, \tag{1}$$

$$s = \frac{n}{2}(a + l). \tag{2}$$

Eliminating l, we obtain

$$2s = n[a + a + (n - 1)d];$$

collecting, gives

$$n^2d + (2a - d)n - 2s = 0.$$

Solving for n, we have

$$n = \frac{-(2a - d) \pm \sqrt{(2a - d)^2 - 4d(-2s)}}{2d}.$$

substituting $a = 3$, $d = 4$, and $s = 300$, gives

$$n = \frac{-(6 - 4) \pm \sqrt{(6 - 4)^2 - 16(-600)}}{8}$$

$$= \frac{-2 \pm \sqrt{9{,}604}}{8}$$

$$= \frac{-2 \pm 98}{8};$$

hence,

$$n = 12 \quad \text{or} \quad n = -\tfrac{25}{2}.$$

Since it is possible to express each of the five letters of an arithmetic progression in terms of any three letters, there exist four distinct formulas for each of the five letters, or twenty formulas in all. Hence, *with the two given formulas* (1) *and* (2) *it is possible to derive eighteen other distinct formulas for arithmetic progressions.*

All the terms of an arithmetic progression between the first term and the last term are called the **arithmetic means** between those

two terms. To insert k arithmetic means between two given numbers, it is necessary to find the common difference d. The common difference d can be found by letting a equal one of the given numbers, l the other given number, and n equal $k + 2$. Then after substituting these values in Equation (1), solve the resulting equation for d. Knowing the common difference, the k means can be found. To illustrate:

Example 3. Insert 5 arithmetic means between 2 and 20.

Solution. The progression will contain 7 terms, the 5 means and the two end terms. Thus, substituting $a = 2$, $l = 20$, and $n = 7$ in Equation (1), we have

$$20 = 2 + (7 - 1)d.$$

Solving for d gives

$$d = 3.$$

Hence, the progression is 2, **5, 8, 11, 14, 17,** 20, and the five means are

5, 8, 11, 14, and 17.

EXERCISES

1. Given 3, 8, 13, 18, · · · to 15 terms; find l and s.
2. Given $-9, -6, -3, 0, \cdots$ to 25 terms; find l and s.
3. Given $-2, -5, -8, -11, \cdots$ to 16 terms; find l and s.
4. Given $3, 5\frac{1}{2}, 8, 10\frac{1}{2}, \cdots$ to 18 terms; find l and s.
5. Given $\frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{11}{6}, \cdots$ to 20 terms; find l and s.
6. Given $a = 8$, $d = -3$, $l = -52$; find n and s.
7. Given $a = -3$, $l = 37$, $s = 85$; find n and d.
8. Given $a = 5$, $n = 5$, $s = 5$; find l and d.
9. Given $n = 10$, $l = 10$, $d = 10$; find a and s.
10. Given $s = 55$, $l = 20$, $d = 3$; find a and n.
11. Express l in terms of a, n, and s.
12. Express l in terms of a, s, and d.
13. Express l in terms of n, s, and d.
14. In the progression a, $(a + d)$, $(a + 2d)$, $(a + 3d)$, · · · , what is the 10th term? 100th term? rth term? $(n + 1)$st term? $(n - 5)$th term?
15. If the first term of an A.P. is -10 and the 10th term is 10, find the 19th term.
16. If the 6th term of an A.P. is 1 and the 12th term is 8, find the 20th term.
17. Find the sum of the first 20 integers; of the first 100 integers.
18. Insert 7 arithmetic means between 10 and 20.
19. Insert 10 arithmetic means between 30 and 5.

20. Insert 8 arithmetic means between -2 and -1.

21. Insert 6 arithmetic means between $\frac{1}{3}$ and $\frac{3}{2}$.

22. What is the formula for the sum of the first n integers?

23. What is the formula for the sum of the first n odd integers?

24. A clock that strikes the number of the hour will strike how many times a day (24 hours)?

25. Find the sum of all the numbers between 100 and 1,000 ending in 5.

26. How many linear feet of rungs will it take to make a ladder with straight sides if there are 18 rungs, the bottom rung being 30 inches long and the top rung 13 inches long?

27. If a body falls 16.1 feet the first second, 48.3 feet the second second, and 80.5 feet the third second, how far will it fall in 10 seconds?

28. In a potato race there are 20 potatoes placed in a straight line at intervals of 5 feet. If the first potato is 10 feet from the basket, how far will an individual run in gathering up the potatoes if he starts at the basket and carries the potatoes, one at a time, back to the basket?

29. A man owes \$4,000 on a home and agrees to pay \$20 per month on the principal plus the interest on the outstanding principal at the rate of 6 per cent. Thus, his first payment is \$20 principal and \$20 interest. The interest decreases 10 cents per month thereafter. Find the number of payments necessary to pay for the house, and the total sum of money paid.

84. GEOMETRIC PROGRESSIONS

Another simple type of series are those in which each term after the first is formed by multiplying the preceding term by a given number as:

$$2 + 4 + 8 + 16 + \cdots,$$
$$2 + 6 + 18 + 54 + \cdots,$$
$$3 - 6 + 12 - 24 + \cdots.$$

The terms of each such series form a geometric progression; that is, a ***geometric progression*** *is a sequence of numbers, called terms, each of which after the first is obtained by multiplying the preceding term by a fixed number called the* ***common ratio.*** For example, each of the following is a geometric progression

$$2, 4, 8, 16, \cdots,$$
$$2, 6, 18, 54, \cdots,$$
$$3, -6, 12, -24, \cdots.$$

Each term of the first progression, after the first term, is formed by multiplying the preceding term by 2; each term of the second progression, after the first term, is formed by multiplying the

preceding term by 3; and each term of the third progression, after the first term, is formed by multiplying the preceding term by -2.

If a geometric progression is given, *the common ratio is found by taking the ratio of any term after the first to the term preceding it.* In the first progression of the foregoing set, the common ratio is 2; in the second progression, the common ratio is 3; and in the third progression, the common ratio is -2.

Let a = the first term,
r = the common ratio,
n = the number of terms to be considered,
l = the last or the n term, and
s = the sum of the first n term.

If a is the first term of a geometric progression and r is the common ratio, then the second term is ar, the third term is ar^2, the fourth term is ar^3, and so on. Finally the nth term or the last term is

$$l = ar^{n-1}. \tag{3}$$

By the sum of a geometric progression is meant the sum of the first n terms. Hence the sum is the sum of the following series

(I) $$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1}.$$

Multiplying (I) by r gives

(II) $$sr = ar + ar^2 + ar^3 + \cdots + ar^{n-2} + ar^{n-1} + ar^n.$$

Subtracting (II) from (I) gives

$$s - sr = a - ar^n;$$

factoring, we have

$$s(1 - r) = a(1 - r^n).$$

Solving for s gives

$$s = \frac{a(1 - r^n)}{1 - r} \quad \text{or} \quad s = \frac{a(r^n - 1)}{r - 1}. \tag{4}$$

To avoid negative numbers, use the first form for $r < 1$ and use the second form for $r > 1$.

The two formulas given by Equations (3) and (4) are sufficient to solve completely most of the elementary problems arising in geometric progressions. Each of the equations (3) and (4) contains four letters, and the two equations together contain all five letters. Therefore, *if the values of three letters of a geometric progression are given, it is possible to solve for the values of the other two*

letters by substituting the values of the three given letters in Equations (3) *and* (4), *and then solving the resulting two equations as a system of two equations in two unknowns.*

Before attempting to solve any problem in geometric progression, it is necessary to ascertain the values of three of the five letters. To illustrate:

Example 1. Find the sixth term and the sum of the first six terms of the geometric progression $2, -6, 18, -54, \cdots$.

Solution. In this example $a = 2$, $r = -3$, and $n = 6$. We substitute these values in Equations (3) and (4), and find the sixth term

$$l = 2(-3)^{6-1} = 2(-243) = -486,$$

and the sum of the first six terms is

$$s = \frac{2[1 - (-3)^6]}{1 - (-3)} = \frac{2(1 - 729)}{1 + 3} = \frac{-1{,}456}{4} = -364.$$

Example 2. Given $r = 2$, $l = 96$, and $s = 189$. Find a and n.

Solution. Substituting these values in Equations (3) and (4), we have

$$\text{(I)} \qquad 96 = a2^{n-1},$$

$$\text{(II)} \qquad 189 = \frac{a(2^n - 1)}{2 - 1} = a2^n - a.$$

To eliminate n, multiply the members of Equation (I) by 2, and subtract Equation II, as follows

$$\begin{array}{rl} 192 = & a \cdot 2^n \\ 189 = & a \cdot 2^n - a \\ \hline 3 = & a \end{array}$$

Hence,

$$a = 3.$$

Substituting this value of a in Equation (I) to solve for n, we have

$$96 = 3 \cdot 2^{n-1},$$

or

$$2^{n-1} = 32 = 2^5.$$

Hence,

$$n - 1 = 5,$$

or

$$n = 6.$$

It also is possible to solve Example 2 by first deriving a formula expressing a in terms of the three given letters r, l, and s. This formula can

be found by eliminating n between Equations (3) and (4). These two equations are

$$l = ar^{n-1}, \tag{3}$$

$$s = \frac{a(r^n - 1)}{r - 1} = \frac{ar^n - a}{r - 1}. \tag{4}$$

Multiplying Equation (3) by r, we obtain

$$rl = ar^n;$$

substituting $ar^n = rl$ in Equation (4) gives

$$s = \frac{rl - a}{r - 1};$$

solving for a gives

$$a = rl - s(r - 1).$$

Substituting $r = 2$, $l = 96$, and $s = 189$, we have

$$a = 2 \cdot 96 - 189(2 - 1),$$

hence,

$$a = 3.$$

All the terms of a geometric progression between the first term and the last term are called **geometric means** between those two terms. To insert k geometric means between two given numbers, it is necessary to find the common ratio r. The common ratio r can be found by letting a equal one of the given numbers, l the other number, and n equal $k + 2$. Then after substituting these values in Equation (3), we solve the resulting equation for r. Knowing the common ratio, we can find the k means. To illustrate:

Example 3. Insert 5 geometric means between $\frac{1}{18}$ and $\frac{81}{2}$.

Solution. The progression will contain 7 terms, the 5 means and the two end terms. Thus, substituting $a = \frac{1}{18}$, $l = \frac{81}{2}$, and $n = 7$ in Equation (3), we have

$$\frac{81}{2} = \frac{1}{18} r^6 \qquad \text{or} \qquad \frac{3^4}{2} = \frac{1}{2 \cdot 3^2} r^6,$$

whence,

$$r^6 = 3^4 \cdot 3^2 = 3^6,$$

and

$$r = 3.$$

Hence, the progression is $\frac{1}{18}, \frac{1}{6}, \frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{27}{2}, \frac{81}{2}$, and the five means are

$$\tfrac{1}{6}, \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{9}{2}, \tfrac{27}{2}.$$

85. GEOMETRIC PROGRESSION WITH INFINITELY MANY TERMS

Let us consider the sum of the infinite series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots .$$

This sum can be shown geometrically by the aid of a line AB which is 2 units long. See Fig. 42. Divide the line AB in the center at P_1, divide P_1B in the center at P_2, divide P_2B in the center at P_3, and so on. It is evident from Fig. 42 that the sum of a finite number of terms of the series

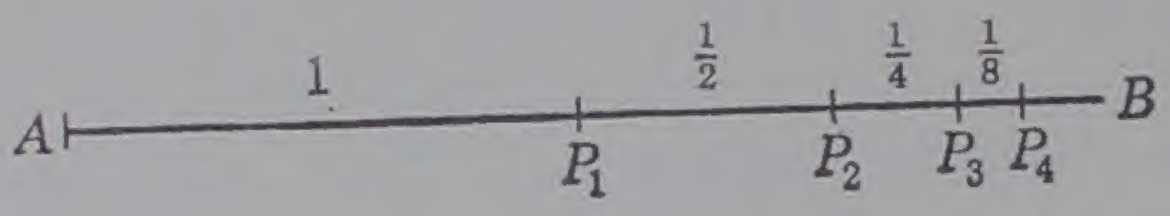


FIG. 42.

$$AP_1 + P_1P_2 + P_2P_3 + P_3P_4 + \cdots ,$$

or

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots$$

cannot exceed nor equal 2, no matter how many terms are taken. However, it should be observed that the sum of a sufficiently large number of terms can be taken so that this sum is as close to 2 as we please.

If s_n represents the sum of the first n terms of a geometric series, then we define the sum of a geometric progression with infinitely many terms to be the limit of s_n as n increases without bound. Thus, we say that the sum of the series given in the preceding paragraph is 2, and we write

$$2 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots . \tag{5}$$

The sum of the first n terms of any geometric progression is given by Equation (4), and thus is

$$s_n = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}. \tag{6}$$

From the fact that any integral power of a number whose numerical value is less than 1 is numerically less than the number itself, it follows that if r is numerically less than 1, r^n, and hence $\dfrac{ar^n}{1 - r}$, in Equation (6) can be made as small as we please by taking n sufficiently large. Hence,

$$s = \lim_{n \to \infty} s_n = \frac{a}{1 - r}.$$

Therefore the sum of a geometric progression with infinitely many terms and with ratio numerically less than 1 is

$$s = \frac{a}{1 - r}. \tag{7}$$

Example 1. Find the sum of the infinite series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$.

Solution. In this case $a = 1$ and $r = \frac{1}{2}$, hence,

$$s = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{2}} = 2.$$

Geometric progressions with infinitely many terms can be applied to finding the proper fraction that is equivalent to a given repeating decimal. To illustrate:

Example 2. What proper fraction is equivalent to the repeating decimal $0.44444 \cdots$?

Solution. This repeating decimal may be written as follows:

$$0.44444 \cdots = 0.4 + 0.04 + 0.004 + 0.0004 + \cdots,$$

which is an infinite series whose terms form an infinite geometric progression in which $a = 0.4$ and $r = 0.1$. Hence the equivalent fraction is

$$s = \frac{a}{1 - r} = \frac{0.4}{0.9} = \frac{4}{9};$$

that is,

$$\tfrac{4}{9} = 0.44444 \cdots.$$

If not all the integers repeat from the first, then form a geometric series by using only the repeating integers. The equivalent fraction is found by adding the nonrepeating integers to the sum of the infinite geometric series. To illustrate:

Example 3. What fraction is equivalent to the repeating decimal $1.2343434 \cdots$?

Solution. This repeating decimal can be written as

$$1.2343434 \cdots = 1.2 + (0.034 + 0.00034 + 0.0000034 + \cdots)$$

in which $a = 0.034$ and $r = 0.01$. Then

$$\begin{aligned} 1.2343434 \cdots &= 1.2 + \frac{0.034}{0.99} \\ &= \tfrac{12}{10} + \tfrac{34}{990} \\ &= \tfrac{611}{495}, \end{aligned}$$

which is the equivalent fraction. That is,

$$\tfrac{6\,1\,1}{4\,9\,5} = 1.2343434 \cdots .$$

However, a shorter method for finding the sum of an infinite series whose terms form a geometric progression is to use the method employed in deriving Formula (4); that is, to multiply both members of the given equation either by r or by $\frac{1}{r}$ and then subtract the members of the two equations. To illustrate:

In Example 1, let

$$S = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \cdots ,$$

Multiplying by $\frac{1}{2}$ gives

$$\tfrac{1}{2}S = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \cdots ,$$

subtracting, we have

$$\tfrac{1}{2}S = 1,$$

hence,

$$S = 2.$$

In Example 2, let

$$S = 0.444444 \cdots .$$

Multiplying by .1 gives

$$.1S = 0.044444 \cdots ,$$

subtracting, we have

$$.9S = 0.4,$$

hence,

$$S = \frac{0.4}{0.9} = \frac{4}{9}.$$

We could have multiplied

$$S = 0.444444 \cdots$$

by 10, giving

$$10S = 4.444444 \cdots ;$$

subtracting $S = 0.444444 \cdots$, we have

$$9S = 4,$$

hence,

$$S = \frac{4}{9}.$$

In Example 3, let

$$S = 1.2343434 \cdots .$$

Multiplying by 100, we have

$$100S = 123.4343434 \cdots;$$

subtracting $S = 1.2343434 \cdots$ gives

$$99S = 122.2,$$

hence,

$$S = \frac{122.2}{99} = \frac{1,222}{990} = \frac{611}{495}.$$

EXERCISES

1. Given 2, 4, 8, 16, $\cdots$ to 9 terms; find l and s.
2. Given 1, -3, 9, -27, $\cdots$ to 8 terms; find l and s.
3. Given 4, 2, 1, $\frac{1}{2}$, $\cdots$ to 10 terms; find l and s.
4. Given -3, 2, $-\frac{4}{3}$, $\frac{8}{9}$, $\cdots$ to 9 terms; find l and s.
5. Given 4, 0.4, 0.04, 0.004, $\cdots$ to 10 terms; find l and s.
6. Given $a = 3$, $r = 2$, $n = 7$; find l and s.
7. Given $n = 10$, $r = 3$, $l = 1{,}458$; find a and s.
8. Given $a = 5$, $n = 4$, $s = 200$; find l and r.
9. Given $n = 6$, $a = 100$, $l = 0.001$; find r and s.
10. Given $a = 2$, $r = 3$, $l = 162$; find n and s.
11. Express r in terms of s, a, and l.
12. Express s in terms of r, l, and a.
13. Express l in terms of s, r, and n.
14. Express a in terms of r, l, and s.
15. In the progression a, ar, ar^2, ar^3, $\cdots$, what is the 12th term? 19th term? kth term? $(k+1)$st term? $(n-5)$th term?
16. If the first term of a G.P. is 3 and the 5th term is 48, find the 8th term.
17. If the 5th term of a G.P. is 1 and the 8th term is $\frac{1}{8}$, find the first term.
18. Given $a = 18$, $l = \frac{3}{9}\frac{2}{}$, $s = \frac{4}{9}\frac{2}{}\frac{2}{}$; find n and r.
19. Insert two geometric means between 6 and 750.
20. Insert five geometric means between 8 and $\frac{1}{8}$.
21. Insert seven geometric means between 1 and 2.
22. Insert four geometric means between $\frac{9}{4}$ and $\frac{8}{27}$.
23. Insert five geometric means between a and b.
24. What is the 10th term of the progression a^2, ab, b^2?
25. A safety campaign was launched in which each individual receiving a letter was to send a similar letter to 10 friends urging them to observe the principles of safety. Suppose the letters were mailed regularly every Monday for 10 weeks beginning with a single individual. What would be the total cost of postage if everyone did his part? (Assume 2-cent postage.)

26. To raise money, it was suggested that bridge parties be held each week, each party consisting of three tables, and that each guest should contribute \$1. Each guest at the first party was to give a party of three tables, and so on. What sum of money would be raised at the end of seven weeks? (Assume that each hostess plays.)

27. If 1 grain of wheat is placed on the first square of a checkerboard, 2 grains on the second square, 4 grains on the third square, 8 grains on the fourth square, and so on, give the order of the number representing the number of grains of wheat necessary to cover the checker board in this manner.

28. If the volume of the cylinder of an air pump is 10 cubic feet and the volume of the connecting tank is 100 cubic feet, what per cent of air remains in the tank after 6 strokes? After 10 strokes?

29. A certain culture of bacteria doubles the number of bacteria every 2 hours. At the end of the day (24 hours) the number of bacteria will be how many times the original culture?

30. If the average number of descendants in each family is 4, and the average length of each generation is 30 years, find how many times the population will multiply itself in 600 years.

Find the sum of each of the following infinite series (Exercises 31 through 38).

31. $12 + 6 + 3 + \frac{3}{2} + \cdots$.

32. $4 - 3 + \frac{9}{4} - \frac{27}{16} + \cdots$.

33. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$.

34. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$.

35. $1 + 0.1 + 0.01 + 0.001 + \cdots$.

36. $0.16 + 0.0016 + 0.000016 + \cdots$.

37. $1 + x + x^2 + x^3 + x^4 + \cdots$ $(x < 1)$.

38. $1 - x + x^2 - x^3 + x^4 - \cdots$ $(x < 1)$.

Reduce each of the following repeating decimals to proper fractions.

39. $0.222222 \cdots$.

40. $0.434343 \cdots$.

41. $0.555555 \cdots$.

42. $0.0222222 \cdots$.

43. $0.0199999 \cdots$.

44. $0.1454545 \cdots$.

45. $1.343434 \cdots$.

46. $0.123123 \cdots$.

47. $0.855555 \cdots$.

48. A rubber ball falls 15 feet and at each fall rebounds $\frac{3}{4}$ of the distance it fell. How far will the ball travel before coming to rest?

49. A pendulum gradually comes to rest as a result of friction. If on each upswing the length of the arc through which the pendulum swings is 0.99 of the arc through which it fell, find the distance traveled before the pendulum comes to rest if the initial arc was 25 inches.

86. HARMONIC PROGRESSIONS

A harmonic progression *is a sequence of numbers whose reciprocals form an arithmetic progression.* To illustrate, the sequence of

numbers

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \cdots$$

form a harmonic progression, since their reciprocals 1, 2, 3, 4, $\cdots$ form an arithmetic progression.

Example 1. Find the seventh term of the harmonic progression

$$3, 2, \tfrac{3}{2}, \cdots .$$

Solution. The reciprocals of these terms form the arithmetic progression $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \cdots$, whose seventh term can be found by substituting $a = \frac{1}{3}$, $n = 7$, and $d = \frac{1}{6}$ in the formula

$$l = a + (n - 1)d.$$

Substituting, we have

$$l = \tfrac{1}{3} + (7 - 1)\tfrac{1}{6} = \tfrac{4}{3}.$$

Hence the seventh term of the harmonic progression is $\frac{3}{4}$.

All the terms of a harmonic progression between the first term and the last term are called the **harmonic means** between those terms. To insert k harmonic means between any two given numbers, find the k arithmetic means between the reciprocals of the two numbers. The reciprocals of the arithmetic means are the required harmonic means.

Example 2. Insert six harmonic means between 1 and 8.

Solution. In the corresponding arithmetic progression $a = 1$, $l = \frac{1}{8}$, and $n = 8$. Hence,

$$l = a + (n - 1)d.$$

Substituting, we get

$$\tfrac{1}{8} = 1 + 7d;$$

then

$$d = -\tfrac{1}{8}.$$

Hence the six arithmetic means between 1 and $\frac{1}{8}$ are $\frac{7}{8}, \frac{3}{4}, \frac{5}{8}, \frac{1}{2}, \frac{3}{8}$, and $\frac{1}{4}$. Therefore the six harmonic means between 1 and 8 are

$$\tfrac{8}{7}, \tfrac{4}{3}, \tfrac{8}{5}, 2, \tfrac{8}{3}, 4.$$

EXERCISES

Give the next four terms in each of the following harmonic progressions.

1. $6, 4, 3, \cdots$.

2. $\frac{3}{2}, \frac{3}{4}, \frac{1}{2}, \cdots$.

3. $\frac{4}{3}, 1, \frac{4}{5}, \cdots$.

4. $\frac{2}{3}, 1, 2, \cdots$.

5. $\frac{1}{4}, 1, -\frac{1}{2}, \cdots$.

6. $2, \frac{16}{9}, \frac{8}{5}, \cdots$.

7. $3, \frac{6}{7}, \frac{1}{2}, \cdots$.

8. $\frac{3}{2}, 2, 3, \cdots$.

9. Insert 3 harmonic means between 1 and 2.

10. Insert 5 harmonic means between $\frac{1}{2}$ and $\frac{1}{3}$.

11. Insert 4 harmonic means between 5 and $\frac{2}{5}$.

12. Insert 7 harmonic means between $\frac{2}{3}$ and $\frac{5}{6}$.

13. Insert 5 harmonic means between $\frac{1}{8}$ and $\frac{3}{4}$.

14. Insert 9 harmonic means between $\frac{2}{3}$ and $\frac{3}{2}$.

15. If a sequence of numbers forms a harmonic progression, show that the sequence of numbers formed by multiplying each member of the given sequence by a constant also forms a harmonic progression.

16. If A, G, and H stand respectively for the arithmetic, geometric, and harmonic means between the two numbers a and b, show that $G^2 = AH$.

17. If a^2, b^2, and c^2 form an arithmetic progression, show that $(b + c)$, $(a + c)$, and $(a + b)$ form a harmonic progression.

18. If a, b, and c form an arithmetic progression, and b, c, and d form a harmonic progression, prove that $ad = bc$.

CHAPTER XV

Permutations, Combinations, and Probability

87. PERMUTATIONS

The study of permutations and combinations is based upon the two following fundamental principles.

(I). *If one thing can occur in* m *different ways, and after it has happened in any of these ways a second thing can occur in* n *different ways, then both events can occur in the order stated, in* $m \cdot n$ *different ways* (m times n ways). To illustrate:

Example 1. How many whole numbers, each less than 5,000 and each containing four different digits, can be formed with the digits 1, 2, 3, 4, 5, 6, 7, 8?

Solution. If the numbers are less than 5,000, then the digit in the thousands place must be chosen from the digits 1, 2, 3, and 4; that is, the first digit can be selected in but 4 ways. After the first digit has been selected the second digit can be selected in 7 ways from the remaining 7 digits, the third digit can be selected in 6 ways, and the fourth in 5 ways. Hence, the total number of whole numbers, each less than 5,000 and each containing four different digits, that can be formed with the digits 1, 2, 3, 4, 5, 6, 7, 8 is

$$4 \cdot 7 \cdot 6 \cdot 5 = 840.$$

(II). *If one of two independent events can occur in* m *different ways, and the other in* n *different ways, then both can occur in* $m \cdot n$ *different ways.* To illustrate:

Example 2. A man has 3 pairs of shoes, 5 suits, and 2 hats. How many outfits, all different, can he select?

Solution. In this case a selection of a suit is independent of a selection of a pair of shoes or a hat and the order of selection is immaterial. The pair of shoes can be selected in 3 ways, the suit in 5 ways, and the hat in 2 ways. Hence, the total number of outfits, all different, is

$$3 \cdot 5 \cdot 2 = 30.$$

Example 3. How many three-digit numbers can be formed from the five digits 1, 2, 3, 4, 5, no digit being repeated in any one number?

Solution. Since any of the five digits can be used for the first digit in the given number, and after the first digit has been written any of the remaining four digits can be used for the second digit in the number, and after the first two digits have been written, any of the remaining three digits can be used for the third digit, it follows that the number of three-digit numbers that can be formed with the digits 1, 2, 3, 4, 5 is

$$5 \cdot 4 : 3 = 60,$$

and the different numbers are given in the accompanying table.

123	124	125	134	135	145	234	235	245	345
132	142	152	143	153	154	243	253	254	354
213	214	215	314	315	415	324	325	425	435
231	241	251	341	351	451	342	352	452	453
312	412	512	413	513	514	423	523	524	534
321	421	521	431	531	541	432	532	542	543

Each of these numbers represents an arrangement of the five digits. Each of these arrangements is called a **permutation.** *A permutation of* r *things is an arrangement of these* r *things in some particular order.* The total number of permutations of n things taken r at a time is the total number of distinct arrangements of r things at a time that can be formed from the n things. The symbol ${}_nP_r$ is used to represent the number of permutations of n things taken r at a time.

In Example 3, the number of permutations of 5 things taken 3 at a time can now be written as

$${}_5P_3 = 5 \cdot 4 \cdot 3 = 60.$$

To determine the number of permutations of n things taken r at a time, where n and r are positive integers and $n \geqq r$, observe that the first place in the arrangement or permutation can be filled in n ways. After the first place has been filled there are left $n - 1$ things to choose from to fill the second place. After the first two places have been filled there are left $n - 2$ things to choose from to fill the third place. After the first three places have been filled there are left $n - 3$ things to choose from to fill the fourth place, and so on. Finally, after the first $r - 1$ places have been filled there are left $n - (r - 1)$ or $n - r + 1$ things to choose from to fill the rth place. Hence, the r places can be filled in $n(n - 1)(n - 2)(n - 3) \cdots (n - r + 1)$ ways, and we have the following theorem:

THEOREM I. *The total number of permutations of* n *different things taken* r *at a time is*

$$ {}_nP_r = n(n-1)(n-2)(n-3)\cdots(n-r+2)(n-r+1). \quad (1) $$

It should be observed that the total number of permutations of n different things taken r at a time consists of the product of r consecutive integers beginning with n and counting backward. To illustrate:

$${}_5P_3 = 5\cdot 4\cdot 3 = 60. \qquad {}_{10}P_4 = 10\cdot 9\cdot 8\cdot 7 = 5{,}040.$$
$${}_5P_4 = 5\cdot 4\cdot 3\cdot 2 = 120. \qquad {}_{13}P_2 = 13\cdot 12 = 156.$$
$${}_5P_5 = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120. \qquad {}_9P_6 = 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4 = 60{,}480.$$

We can also think of the number of permutations of n different things taken r at a time as being the product of r consecutive integers beginning with n and ending with $n-r+1$. Thus, ${}_9P_6$ is the product of six consecutive integers beginning with 9 and ending with $9-6+1=4$. Hence, by filling in with the integers between 9 and 4, we have

$$ {}_9P_6 = 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4 = 60{,}480. $$

If n and r are equal, we have that *the total number of permutations of* n *different things taken* n *at a time is the product of the first* n *integers*

$$ {}_nP_n = n(n-1)(n-2)(n-3)\cdots 4\cdot 3\cdot 2\cdot 1 = n!. \quad (2) $$

It is customary to use the symbol $n!$, read "*factorial* n," to represent the product of the first n integers. Thus, we have

$$ n! = 1\cdot 2\cdot 3\cdot 4\cdots(n-3)(n-2)(n-1)n. \quad (3) $$

To illustrate:

$$3! = 1\cdot 2\cdot 3 = 6,$$
$$5! = 1\cdot 2\cdot 3\cdot 4\cdot 5 = 120,$$
$$7! = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7 = 5{,}040.$$

Sometimes the total number of specified arrangements is found by taking a sum or difference of the number of permutations. To illustrate:

Example 4. How many ways can 5 different books be placed on a shelf so that two given books will *not* be adjacent to each other?

Solution. This problem may be solved in two ways.

First Method. Perhaps the easiest way to solve this problem is to think of the total number of ways 5 books can be placed on a shelf and then to subtract the number of ways two given books can be adjacent to each other. The 5 books can be placed on a shelf in ${}_5P_5 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ different ways. To determine the number of ways the two given books can be adjacent, we can think of the two given books as being tied together; then there are 4 units to be arranged on the shelf. These 4 units can be placed on the shelf in ${}_4P_4 = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways, and since we can interchange the two given books, we have that these two given books can be adjacent to each other $2 \cdot {}_4P_4 = 48$ times out of the possible 120 different ways the five books can be placed on the shelf. Hence, the total number of ways 5 different books can be placed on a shelf so that two given books will *not* be adjacent to each other is the difference

$$ {}_5P_5 - 2{}_4P_4 = 120 - 48 = 72.$$

Solution. *Second Method.* Let A and B be the two given books. The number of different ways the remaining four books can be placed on the shelf so that B is not adjacent to A when A occupies different positions may be shown as follows:

Positions					
1	2	3	4	5	
A	3 $\cdot$	3 $\cdot$	2 $\cdot$	1	= 18 ways
3	A	2 $\cdot$	2 $\cdot$	1	= 12 ways
2 $\cdot$	3	A	2 $\cdot$	1	= 12 ways
2 $\cdot$	1 $\cdot$	3	A	2	= 12 ways
3 $\cdot$	2 $\cdot$	1 $\cdot$	3	A	= 18 ways
				Total	= 72 ways

88. PERMUTATIONS OF THINGS SOME OF WHICH ARE ALIKE

Consider the permutations of n things taken all at a time when the n things are not all distinct. Since the n things are not all distinct, an interchange of two like things will not give essentially a new arrangement or permutation. For example, suppose we wish to find the number of distinct five-digit numbers that can be formed by using the five digits 1, 1, 2, 2, and 2. If the five digits were all distinct there would be ${}_5P_5 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ different numbers, but if we use the five digits 1, 1, 2, 2, and 2, then the 120 numbers are not all distinct, for interchanging the 1's will not change the number, and interchanging the 2's will not change the number. Since the two 1's can be interchanged in

${}_2P_2 = 2 \cdot 1 = 2$ ways without changing the number, and the three 2's can be interchanged in ${}_3P_3 = 3 \cdot 2 \cdot 1 = 6$ ways without changing the number, it follows that in the 120 permutations of the five digits 1, 1, 2, 2, and 2, each number is repeated twice because of the two 1's, and six times because of the three 2's. Hence, the total number of distinct numbers that can be formed with the digits 1, 1, 2, 2, and 2, is

$$\frac{\overset{2}{\cancel{5}} \cdot \overset{2}{\cancel{4}} \cdot \overset{3}{\cancel{3}} \cdot \overset{1}{\cancel{2}} \cdot \overset{1}{\cancel{1}}}{\underset{1}{\cancel{1}} \cdot \underset{1}{\cancel{2}} \cdot \underset{1}{\cancel{1}} \cdot \underset{2}{\cancel{2}} \cdot \underset{3}{\cancel{3}}} = 10,$$

and the ten numbers are

11222, 12122, 12212, 12221, 21122,
21212, 21221, 22112, 22121, 22211.

We shall next find the number of permutations of n things taken n at a time when the n things are not all different. The total number of permutations of n things taken n at a time is ${}_nP_n$ if the n things are all different. If in these ${}_nP_n$ permutations we make $n_1 (n_1 < n)$ of the things alike, then each permutation will be repeated ${}_{n_1}P_{n_1}$ times; because, for a given arrangement of the n things the n_1 like things can be permuted in ${}_{n_1}P_{n_1}$ different ways without essentially changing the arrangement or permutation. If we now make n_2 other things alike, then each permutation will be repeated ${}_{n_2}P_{n_2}$ times. By continuing in this manner for each set of like things, we have the following theorem:

THEOREM II. *The total number of permutations* $\boldsymbol{P}$ *of* $\boldsymbol{n}$ *things taken* $\boldsymbol{n}$ *at a time when* $\boldsymbol{n_1}$ *things are alike,* $\boldsymbol{n_2}$ *other things are alike,* $\boldsymbol{n_3}$ *other things are alike, and so on is*

$$\boldsymbol{P = \frac{{}_nP_n}{{}_{n_1}P_{n_1} \cdot {}_{n_2}P_{n_2} \cdot {}_{n_3}P_{n_3} \cdots} = \frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdots},} \qquad (4)$$

where $\boldsymbol{n = n_1 + n_2 + n_3 + \cdots}$.

Example. How many permutations can be made of the letters in the word *bookkeeper* taken all at a time?

Solution. There are 10 letters in all, 2 *o*'s, 2 *k*'s, 3 *e*'s, and *b*, *p*, and *r*. Hence, the total number of permutations is

$$P = \frac{10!}{2!\,2!\,3!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot \overset{4}{\cancel{4}} \cdot \overset{3}{\cancel{3}} \cdot \overset{2}{\cancel{2}} \cdot 1}{(1 \cdot \underset{2}{\cancel{2}})(1 \cdot \underset{2}{\cancel{2}})(1 \cdot \underset{2}{\cancel{2}} \cdot \underset{3}{\cancel{3}})} = 205{,}800.$$

QUESTIONS

1. What are the two fundamental principles?
2. What is the formula for the number of permutations of n different things taken r at a time?
3. What is the formula for the number of permutations of n different things taken n at a time?
4. What is the formula for the number of permutations of n things taken n at a time if the n things are not all different?

EXERCISES

1. How many numbers of 5 digits each can be formed by using all the digits 1, 2, 3, 4, and 5?
2. How many numbers of 7 digits each can be formed by using all the digits 1, 2, 3, 4, 5, 6, and 7?
3. How many numbers of 4 digits each can be formed by using four of the digits 1, 2, 3, 4, 5, 6, and 7?
4. How many even numbers consisting of 6 digits can be formed by using all the digits 1, 2, 3, 4, 5, and 6?
5. How many numbers of 5 digits each can be formed with the digits 1, 2, 3 (use repetition)?
6. How many numbers consisting of one or more digits can be formed with the digits 1, 2, 3, 4, and 5, if no digit is used more than once in a given number?
7. How many even numbers consisting of 5 digits each can be formed by using all the digits 1, 2, 3, 4, and 5? How many odd numbers consisting of 5 digits each can be formed?
8. How many 5-digit numbers greater than 34,000 can be formed with the digits 1, 2, 3, 4, and 5, no digit being repeated in any one number?
9. How many numbers of 7 digits each can be formed by using all the digits 1, 1, 1, 2, 2, 2, and 2?
10. How many numbers of five digits each can be formed with the digits 1, 2, 3, 4, and 5, if a digit may be used more than once in a given number?
11. With 2 vowels and 3 consonants, how many distinct words beginning with a vowel and ending with a consonant can be formed by using all five letters?
12. With 5 vowels and 8 consonants, how many distinct words beginning with a vowel and containing 5 distinct letters can be formed?
13. In how many ways can 6 persons stand in line so that A and B will be adjacent to each other?
14. In how many ways can 7 persons stand in line so that two given persons will not stand adjacent to each other?

15. In how many ways can 12 marbles (all different) be divided among 3 boys?

16. In how many ways can 12 books be divided among 4 boys?

17. If there are 5 mail boxes on your way to class, in how many ways can you mail 2 letters?

18. What is the maximum number of distinct telephone numbers of 4 digits each that can be formed?

19. In how many ways can the letters of each of the following words be arranged: *Illinois, Mississippi, Massachusetts?*

20. How many distinct words can be formed by using all the letters in the word *Tennessee?* In the word *Cincinnati?*

21. A coach has 12 men on a basketball squad of which 2 men can play center, 6 men can play foreward, and 4 men can play guard. In how many ways can the coach assign the positions on the basketball team?

22. How many words can be formed by using all the letters in the word *Pennsylvania* if the position of the vowels is not changed?

23. Is it possible to make a telegraphic code for the twenty-six letters of our alphabet and the ten digits, if no letter or digit is to contain more than 3 dots and 2 dashes?

24. In how many ways can 4 men and 4 women stand in line so that no men are standing side by side nor any women are standing side by side?

25. In how many ways can 2 dice fall?

26. In how many ways can two dice fall to give the sum 2? The sum 3? the sum 4 and so on? Finally the sum 12?

Write the following in product form.

27. ${}_nP_5 = ?$ **28.** ${}_nP_7 = ?$ **29.** ${}_nP_{r-1} = ?$ **30.** ${}_nP_{r+2} = ?$

31. ${}_{n-3}P_3 = ?$ **32.** ${}_{n-1}P_r = ?$ **33.** ${}_{40}P_{37} = ?$ **34.** ${}_nP_{n-2} = ?$

Find the value of each of the following.

35. $\frac{{}_8P_8}{{}_6P_6}$. **36.** $\frac{{}_{12}P_{12}}{{}_9P_9}$. **37.** $\frac{8!}{7!}$. **38.** $\frac{(n+1)!}{(n-1)!}$.

39. $\frac{6!\,9!}{7!\,8!}$. **40.** $\frac{n!}{(n-1)!}$. **41.** $n! - (n-1)!$. **42.** $(n+1)! - (n-1)!$.

89. COMBINATIONS

Combinations differ from permutations in that a permutation is an arrangement with regard to order in the arrangement, and a *combination is a set or a group of things without regard to any order in the group.* Thus, r things taken r at a time can give only one group or combination, but these r things can be arranged in ${}_rP_r = r!$ permutations, and conversely, all the $r!$ different permutations of r things taken r at a time give the same combination.

The symbol ${}_nC_r$ is used to denote the total number of combinations of n things taken r at a time. We shall illustrate by the following example.

Example 1. Find the number of different committees of 3 men each that can be formed from 5 men.

Solution. In this case we are not interested in any order or arrangement of the men composing a committee. We are merely interested in the number of distinct combinations or committees of 3 men each that can be formed from the 5 men. The 3 men can be selected from 5 men in ${}_5P_3 = 5 \cdot 4 \cdot 3$ different ways. Since the 3 men on each committee can be selected in ${}_3P_3 = 3 \cdot 2 \cdot 1$ different ways, the number of distinct committees is

$$ {}_5C_3 = \frac{{}_5P_3}{{}_3P_3} = \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \overset{\cdot}{3}}{1 \cdot 2 \cdot \cancel{3}} = 10. $$

The 60 permutations of 5 things taken 3 at a time are given in Article 87. It should be observed that if we think of the men as being numbered, there are 6 permutations giving the same combination or committee. Thus, we may suppose that the 10 different committees are indicated by the first row in the arrangement given in Article 87.

Since there are ${}_nP_r$ distinct permutations of n things taken r at a time, and there are ${}_rP_r = r!$ permutations that have the same combination, we have the following theorem:

THEOREM I. *The total number of combinations of n things taken r at a time is the number of permutations of n things taken r at a time divided by the number of permutations of r things taken r at a time.* Therefore,

$$ {}_nC_r = \frac{{}_nP_r}{{}_rP_r} = \frac{{}_nP_r}{r!} $$

$$ = \frac{n(n-1)(n-2)(n-3)\cdots(n-r+3)(n-r+2)(n-r+1)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (r-2)(r-1)(r)} \qquad (5) $$

Observe that the number of combinations of n things taken r at a time is a quotient whose denominator is factorial r, representing the product of the first r integers, and the numerator is the product of r integers beginning with n and decreasing consecutively. To find the value of ${}_nC_r$, it is helpful to write the r consecutive integers in the denominator first, then write in the

numerator a number above each number in the denominator, placing n above 1, $n-1$ above 2, and so on. Cancellation should be employed to simplify this quotient, which is always an integer. To illustrate:

$$ {}_7C_4 = \frac{{}_7P_4}{{}_4P_4} = \frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4}}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4}} = 35. $$

$$ {}_{11}C_5 = \frac{{}_{11}P_5}{{}_5P_5} = \frac{11 \cdot \cancel{10} \cdot \cancel{9} \cdot \cancel{8} \cdot 7}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5}} = 462. $$

$$ {}_{10}C_4 = \frac{{}_{10}P_4}{{}_4P_4} = \frac{10 \cdot \cancel{9} \cdot \cancel{8} \cdot 7}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4}} = 210. $$

$$ {}_{10}C_6 = \frac{{}_{10}P_6}{{}_6P_6} = \frac{10 \cdot \cancel{9} \cdot \cancel{8} \cdot 7 \cdot \cancel{6 \cdot 5}}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5 \cdot 6}} = 210. $$

Since the symbol ${}_{10}C_6$ has the same value as the symbol ${}_{10}C_4$, it suggests the following: After r things have been selected to form a combination, the remaining $n-r$ things can form but the one combination. Hence, it follows that *there must be as many combinations of* $\boldsymbol{n}$ *things taken* $\boldsymbol{r}$ *at a time as there are combinations of* $\boldsymbol{n}$ *things taken* $\boldsymbol{n-r}$ *at a time.* This is shown algebraically by

$$ \frac{n(n-1)(n-2)\cdots(r+1) \quad (r)\cdots(n-r+1)}{1 \cdot 2 \quad \cdots \quad (n-r)(n-r+1)\cdots(r)} = \frac{n(n-1)\cdots(r+1)}{1 \cdot 2 \quad \cdots \quad (n-r)}. \tag{6} $$

Hence,

$$ {}_n\boldsymbol{C}_r = {}_n\boldsymbol{C}_{n-r}. \tag{7} $$

Therefore, when finding the value of ${}_nC_r$ we may use the smaller of the two numbers r and $n-r$. To illustrate:

$$ {}_{50}C_{48} = {}_{50}C_2 = \frac{50 \cdot 49}{1 \cdot 2} = 1{,}225. $$

From the fundamental principle, (II), Article 87, we have the following important relation.

If there are several groups of things, the first group containing $\boldsymbol{n_1}$ *things, the second group containing* $\boldsymbol{n_2}$ *things, and so on, then the total number of combinations of* $\boldsymbol{r_1 + r_2 \cdots}$ *things taken at a*

time, of which r_1 *things are from the first group,* r_2 *things from the second group, and so forth, is the product* ${}_{n_1}C_{r_1} \cdot {}_{n_2}C_{r_2} \cdots$ (but never their sum).

Example 2. In how many ways can 3 consonants and 2 vowels be selected from 8 consonants and 5 vowels?

Solution.

$$ {}_8C_3 \cdot {}_5C_2 = \frac{\overset{4}{\cancel{8}} \cdot 7 \cdot \overset{2}{\cancel{6}}}{1 \cdot \cancel{2} \cdot \cancel{3}} \cdot \frac{5 \cdot \overset{2}{\cancel{4}}}{1 \cdot \cancel{2}} = 56 \cdot 10 = 560. $$

The actual computation necessary to solve problems in permutations and combinations is not difficult. The most difficult part of such problems is analyzing the problem to determine whether it is a permutation problem, or a combination problem, or a combination of the two, or whether it is the sum of several permutations or combinations.

Example 3. How many distinct words or symbols composed of 3 different consonants and 2 different vowels each can be formed from thirteen different letters of which 8 are consonants and 5 are vowels (assuming all different arrangements of letters to form words or symbols)?

Solution. In this case it is necessary: First, to determine the number of ways in which 3 consonants can be selected from 8 consonants, and 2 vowels from 5 vowels; second, to determine the number of different words that can be formed with each selection of 3 consonants and 2 vowels. The first part is a combination problem and the second part is a permutation problem.

The 3 consonants can be selected from the 8 consonants in ${}_8C_3 = \frac{8 \cdot 7 \cdot \cancel{6}}{1 \cdot \cancel{2} \cdot \cancel{3}}$ $= 56$ ways, and the 2 vowels can be selected from the 5 vowels in ${}_5C_2$ $= \frac{5 \cdot 4}{1 \cdot 2} = 10$ ways. For any one of the 56 selections of 3 consonants, any one of the 10 selections of 2 vowels can be used. Hence, by the fundamental principle (II) given in Article 87, the 3 consonants and 2 vowels can be selected from the 8 consonants and 5 vowels in $56 \cdot 10$ $= 560$ ways. The rest of the problem is to determine the number of words that can be formed with each selection of 3 constants and 2 vowels. This is a permutation problem and for each selection of 3 consonants and 2 vowels there are ${}_5P_5 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ words. Hence, in all there are ${}_8C_3 \cdot {}_5C_2 \cdot {}_5P_5 = 56 \cdot 10 \cdot 120 = 67{,}200$ words.

Another type of problem is the following.

Example 4. In how many ways can 5 coins fall?

Solution. In this case the number of heads or tails to appear is not specified. Thus, it is necessary to determine the sum of the number of ways in which 5 coins can fall no heads, 1 head, 2 heads, 3 heads, 4 heads, and 5 heads. The 5 coins can fall

$$\begin{aligned}
&\text{no heads in} && && = 1 \text{ way}\\
&\text{1 head in } {}_5C_1 = \frac{5}{1} && && = 5 \text{ ways}\\
&\text{2 heads in } {}_5C_2 = \frac{5\cdot 4}{1\cdot 2} && && = 10 \text{ ways}\\
&\text{3 heads in } {}_5C_3 = \frac{5\cdot 4\cdot 3}{1\cdot 2\cdot 3} && && = 10 \text{ ways}\\
&\text{4 heads in } {}_5C_4 = \frac{5\cdot 4\cdot 3\cdot 2}{1\cdot 2\cdot 3\cdot 4} && && = 5 \text{ ways}\\
&\text{5 heads in } {}_5C_5 = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4\cdot 5} && && = \frac{1 \text{ way}}{32 \text{ ways}}
\end{aligned}$$

The number of ways in which 5 coins can fall is $1 + {}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5 = 32 = 2^5$.

Example 4 can also be solved by using Principle (II), Article 87. Since each coin can fall in 2 ways, then the 5 coins can fall in $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$ ways.

For the general case, let us determine the total number of combinations of n things. This can be done by finding the sum of the number of combinations of n things taken 1 at a time, taken 2 at a time, taken 3 at a time, $\cdots$, and finally taken n at a time. Hence, if C denotes the total number of combinations of n things, we have

$$C = {}_nC_1 + {}_nC_2 + {}_nC_3 + \cdots + {}_nC_{n-2} + {}_nC_{n-1} + {}_nC_n. \qquad (8)$$

By means of the following reasoning, this sum is shown to be $2^n - 1$.

Consider the expression

$$(a + x)^n = (a + x)(a + x)(a + x) \cdots (a+x)(a+x)(a+x). \quad (n \text{ factors}) \qquad (9)$$

By removing parentheses we have terms each of which is composed of the product of n letters, where a letter must be taken from each of the n pairs of parentheses. The product of n a's can be obtained in but one way, hence the term a^n. The product of $(n - 1)$ a's and one x can be obtained in ${}_nC_1$ ways, the number of ways in which

one x can be selected from n x's, hence the term ${}_nC_1a^{n-1}x$. The product of $(n-2)$ a's and two x's can be obtained in ${}_nC_2$ ways, the number of ways in which two x's can be selected from n x's, hence the term ${}_nC_2a^{n-2}x^2$, and so on. Therefore,

$$(a+x)^n = a^n + {}_nC_1a^{n-1}x + {}_nC_2a^{n-2}x^2 + \cdots + {}_nC_{n-1}ax^{n-1} + {}_nC_nx^n. \quad (10)$$

If in Equation (10) we let $a = 1$ and $x = 1$, then

$$(1+1)^n = 2^n = 1 + {}_nC_1 + {}_nC_2 + {}_nC_3 + \cdots + {}_nC_{n-2} + {}_nC_{n-1} + {}_nC_n. \quad (11)$$

Equation (11) reduces to Equation (8) if 1 is subtracted from both members of Equation (11). Hence, *the total number of combinations of* n *things is*

$$C = {}_nC_1 + {}_nC_2 + {}_nC_3 + {}_nC_4 + \cdots + {}_nC_{n-2} + {}_nC_{n-1} + {}_nC_n = 2^n - 1. \quad (12)$$

Observe that if 5 coins fall, they can fall all tails in addition to 1, 2, 3, 4, and 5 heads, respectively. Hence, 5 coins can fall in 2^5 ways instead of $2^5 - 1$ ways.

Many problems, unless carefully stated, can be considered either as permutation problems or as combination problems. To illustrate, suppose the question is stated as follows: In how many ways can 3 cards be drawn from a deck of cards? If the 3 cards are drawn one at a time, the problem is a permutation problem. If the 3 cards are drawn at the same time, the problem is a combination problem. In the problems to follow, we have attempted to state each problem carefully so that it is either a permutation problem or a combination problem, but not both.

QUESTIONS

1. What is the total number of combinations of n things taken r at a time; taken $n - r$ at a time?

2. What is the total number of combinations of n things?

3. What is the total number of combinations when the elements are selected from different groups?

PROBLEMS

1. How many distinct committees of 6 persons each can be selected from 12 persons?

2. How many triangles can be formed by joining 10 points in pairs, no three of which are in the same straight line?

3. How many distinct drawings of 3 cards each can be made from a deck of 52 cards?

4. How many distinct drawings of 3 red cards each can be made from a deck of 52 cards?

5. How many distinct drawings of 3 face cards each can be made from a deck of 52 cards?

6. How many different rounds of mixed bridge consisting of one table each can be played with 5 boys and 5 girls so that both pairs are not the same as in any previous round?

7. A class has 25 members, 10 girls and 15 boys. How many different committees of 3 boys and 2 girls each can be formed?

8. How many committees of 5 men and 3 women each can be selected from 10 men and 8 women?

9. In how many ways can 4 balls, of which 2 are red and 2 are white, be selected from a box containing 5 red and 7 white balls?

10. From a box containing 3 red balls, 4 white balls, and 6 black balls, 3 balls are drawn at once. (a) In how many ways can 1 red, 1 white, and 1 black ball be drawn? (b) In how many ways can 1 red and 2 black balls be drawn? (c) In how many ways can 2 red and 1 white ball be drawn?

11. If there are 10 men in a meeting and each shakes hands with all the others, how many handshakes are there?

12. How many planes are determined by 15 points, no 4 of which lie in the same plane?

13. How many baseball teams of 9 men each can be chosen from 19 players of whom 2 can only catch, 5 can only pitch, 7 can only play in the infield, and 5 can only play in the outfield?

14. How many baseball teams can be formed in Problem 13 if 2 of the pitchers can play any of the positions in the outfield?

15. A class has 25 members, 10 girls and 15 boys. How many committees of 5 each can be selected so that each committee contains at least 2 boys?

16. How many committees of 8 persons each can be selected from 10 men and 8 women so that each committee contains at least 4 men?

17. How many words containing 3 vowels and 5 consonants can be formed from 5 vowels and 8 consonants?

18. How many words containing 2 vowels and 3 consonants can be formed from 5 vowels and 8 consonants?

19. How many different sums can be formed with a cent, a nickel, a dime, a quarter, and a half dollar?

20. With 6 flags, each of a different color, how many different signals can be given by using one or more flags at a time placed vertically on a pole?

21. In how many ways can 6 coins fall, showing at most two heads?

22. In how many ways can 6 coins fall, showing at least two heads?

23. There are 2 tall men on a basketball squad of 10 men. (a) How many teams can be formed containing just one tall man? (b) How many teams can be formed containing both tall men? (c) How many teams can be formed containing a tall man?

24. From a squad of 10 men, how many basketball teams can be formed so that two particular men will not be on the same team?

25. In how many ways can a deck of 52 cards be dealt to four players, each player receiving thirteen cards?

26. A basket contains 5 apples, 8 oranges, and 9 bananas. How many different selections of 5 pieces of fruit can be made?

27. At a mixed bridge party of 3 tables, how many rounds of bridge can be played so that not all partners are the same in any round as they were in any previous round?

28. At a mixed bridge party of 3 tables, how many rounds of bridge can be played so that no partners are repeated in any round?

29. How many words containing 3 vowels and 5 consonants can be formed from 5 vowels and 8 consonants, if each word begins and ends with a consonant?

30. A basket contains 6 red balls, 8 white balls, and 4 blue balls. In how many ways can a selection of 3 balls be made so that: (1) all three are red? (2) all three are white? (3) all three are blue? (4) 2 are red and 1 is white? (5) 2 are red and 1 is blue? (6) 2 are white and 1 is blue? (7) 2 are white and 1 is blue? (8) 2 are blue and 1 is white? (9) 2 are blue and 1 is red? (10) 1 is red, 1 is white, and 1 is blue?

Write the following in ratio form.

31. ${}_nC_3 = ?$ **32.** ${}_nC_8 = ?$ **33.** ${}_{n-2}C_5 = ?$

34. ${}_{n-2}C_r = ?$ **35.** ${}_{n-1}C_{r-1} = ?$ **36.** ${}_{n+3}C_{r-3} = ?$

90. MEANING OF PROBABILITY

The most interesting applications of permutations and combinations are in the field of probabilities, and the more useful applications of probabilities are in the fields of insurance and statistics. In this course, we shall be interested in only a few of the simpler problems in probability, to acquaint the reader with the concept of probability.

By probability of success is meant the proportion of successful events out of many trials. For example, out of many tosses of a coin, 50% of the tosses are likely to be heads, and 50% of the tosses

are likely to be tails. It is customary to use the fraction form to represent probabilities, instead of the per cent notation. Thus, the probability that a coin will fall heads up is $\frac{1}{2}$; that is, a coin can fall heads up 1 way out of the 2 possible ways it can fall. In making a trial we often call the happening of the event in question a **success.**

DEFINITION. *The probability that an event will happen is the ratio of the number of successful ways the event can happen to the total possible number of ways the event can happen or fail.*

It is necessary for all events to be equally likely to happen. Consider, for example, what is the probability that the sun will rise tomorrow? Since the sun will either rise tomorrow or it will not rise, we may argue that the events can happen in two ways, one of which is successful. Hence, we may reason that the probability that the sun will rise tomorrow is $\frac{1}{2}$. However, we know that it is a certainty that the sun will rise. (In this case were we considering things equally likely to happen?)

If we make n trials resulting in s successes and $n - s$ failures, then the probability p that one particular trial of the event will result in a success is given by

$$p = \frac{s}{n}, \tag{13}$$

and the probability q that one particular trial of the event will result in a failure is given by

$$q = \frac{n - s}{n}. \tag{14}$$

The sum of the probabilities of success and of failure is therefore equal to

$$p + q = \frac{s}{n} + \frac{n - s}{n} = 1. \tag{15}$$

Example 1. One card is drawn from a deck of 52 cards. What is the probability that the card is an ace?

Solution. One card can be drawn from the deck in 52 ways and one ace can be drawn in 4 ways. Hence, the probability that one card drawn from a deck is an ace is 4 out of 52, or

$$p = \frac{4}{52} = \frac{1}{13}.$$

Many problems in probabilities can be solved by using any one of three essentially different concepts. For certain problems the probability that an event will happen can be considered as the ratio of two permutations, or the ratio of two combinations, or the product of the probabilities that each successive event will be successful. To illustrate the three methods, let us solve the following example.

Example 2. If 3 cards are drawn from a deck of 52 cards, what is the probability that all 3 cards are black?

Solution 1. If we think of the 3 cards as being drawn one at a time, then the 3 cards can be drawn from the deck of cards in ${}_{52}P_3 = 52 \cdot 51 \cdot 50$ different ways, and the 3 black cards can be drawn from the 26 black cards in ${}_{26}P_3 = 26 \cdot 25 \cdot 24$ ways. Hence, the probability that the three cards drawn are black is the ratio of the two permutations

$$p = \frac{{}_{26}P_3}{{}_{52}P_3} = \frac{26 \cdot 25 \cdot 24}{52 \cdot 51 \cdot 50} = \frac{2}{17}.$$

Solution 2. If we think of the 3 cards as being drawn all at the same time, then the 3 cards can be drawn from the deck of cards in ${}_{52}C_3 = \frac{52 \cdot 51 \cdot 50}{1 \cdot 2 \cdot 3} = 22{,}100$ ways, and the 3 black cards can be drawn from the 26 black cards in ${}_{26}C_3 = \frac{26 \cdot 25 \cdot 24}{51 \cdot 2 \cdot 3} = 2{,}600$ ways. Hence, the probability that the 3 cards drawn are black is the ratio of the two combinations

$$p = \frac{{}_{26}C_3}{{}_{52}C_3} = \frac{\dfrac{26 \cdot 25 \cdot 24}{1 \cdot 2 \cdot 3}}{\dfrac{52 \cdot 51 \cdot 50}{1 \cdot 2 \cdot 3}} = \frac{2{,}600}{22{,}100} = \frac{2}{17}.$$

Solution 3. If we draw one card at a time, then the probability that the first card drawn is a black card is $p_1 = \frac{26}{52}$, there being 26 black cards and 52 cards in all. If the first drawing is successful, only 25 black cards are left in the deck. Hence, the probability that the second card drawn is a black card is $p_2 = \frac{25}{51}$. The probability that the third card drawn is a black card is $p_3 = \frac{24}{50}$. The probability that all three cards drawn are black is the product of the separate probabilities. Hence, we have

$$p = p_1 \cdot p_2 \cdot p_3 = \frac{26}{52} \cdot \frac{25}{51} \cdot \frac{24}{50} = \frac{2}{17}.$$

91. THEOREMS ON PROBABILITY

Theorem for Dependent Events

THEOREM 1. *If the probability of a first event is p_1 and if, after this has happened, the probability of a second event is p_2, then the probability that both events will happen in the order stated is $p_1 p_2$.* (This may be extended to any number of events.)

Proof: If the first event can happen in n_1 ways, and after it has happened the second event can happen in n_2 ways, then from the fundamental principle (I) in Article 87, it follows that the total number of ways in which the two events can happen in the order stated is $n_1 \cdot n_2$. In like manner, if s_1 represents the number of ways in which the first event can be successful, and s_2 the number of ways in which the second event can be successful, then the total number of possible ways in which both events can happen successfully is $s_1 \cdot s_2$. Hence, the probability p that both events are successful in the order stated is

$$p = \frac{s_1 \cdot s_2}{n_1 \cdot n_2} = \frac{s_1}{n_1} \cdot \frac{s_2}{n_2} = p_1 \cdot p_2. \tag{16}$$

Example 1. If 3 cards are drawn from a deck of 52 cards, what is the probability that the 3 cards are spades?

Solution. The probability that the first card drawn is a spade is $p_1 = \frac{13}{52}$. If the first card drawn is successful, the probability that the second card drawn is a spade is $p_2 = \frac{12}{51}$. If the first two cards drawn are spades, the probability that the third card drawn is a spade is $p_3 = \frac{11}{50}$. Therefore, the probability that all three cards drawn are spades is

$$p = p_1 \cdot p_2 \cdot p_3 = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} = \frac{33}{2{,}550}.$$

One should observe the distinction between the probability that each of a set of events will happen and the probability that just one of a set of events will happen. The former is the probability for **independent events** and the latter is the probability for **mutually exclusive events.**

Theorem for Independent Events

THEOREM 2. *The probability that all of a set of independent events will occur is the product of their separate probabilities.*

Example 2. What is the probability of throwing a 2 in each of three successive throws of a single die?

Solution. These events can happen independent of each other. The probability of throwing a 2 in one throw of a die is $\frac{1}{6}$ (a die can fall 6 ways). The probability of throwing a 2 three times in succession is the product of the separate probabilities; that is,

$$p = \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{216}.$$

Theorem for Mutually Exclusive Events

THEOREM 3. *The probability that one of any number of a set of mutually exclusive events will happen in a single trial is the sum of the probabilities for the separate events.*

Proof: If an event can happen in n ways, each equally likely, and if s_1 represents the number of ways in which the event can be successful from the first set and s_2 the number of ways in which the event can be successful from the second set, then the probability p that a single trial will result either in one of the s_1 ways or in one of the s_2 ways is

$$p = \frac{s_1 + s_2}{n} = \frac{s_1}{n} + \frac{s_2}{n} = p_1 + p_2. \tag{17}$$

Example 3. Find the probability of throwing a 1, or a 2, or a 3 in a single throw of one dice.

Solution. Since throwing a 1 with a single throw of a die excludes the throwing of a 2 or a 3, and the same for a 2 or 3, the three events are said to be **mutually exclusive.** The probability of throwing any one of three numbers in a single throw of a die is the sum of the separate probabilities; that is,

$$p = \tfrac{1}{6} + \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{2}.$$

Theorems for Repeated Trials

THEOREM 4. *If $\boldsymbol{p}$ is the probability that an event will happen in any single trial, then the probability $\boldsymbol{P}$ that an event will happen exactly $\boldsymbol{r}$ times in $\boldsymbol{n}$ trials is*

$$\boldsymbol{P = {}_nC_r p^r q^{n-r}} \qquad \text{or} \qquad \boldsymbol{P = {}_nC_{n-r} p^r q^{n-r}}. \tag{18}$$

Proof: To find the probability that an event will happen exactly r times in n trials, the conditions must be the same for each trial. Let p denote the probability that the event will happen in a single

trial and q the probability that it will fail ($q = 1 - p$). The probability that the first r trials are successes and the remaining $n - r$ trials are failures is $p^r q^{n-r}$. By Article 89, the r trials can be selected from n trials in ${}_nC_r$ ways. Thus the r events can happen in ${}_nC_r$ different orders, and the probability that the r events will happen in any one of these orders is ${}_nC_r$ times as great as the probability that the event will happen in a particular order. Hence $P = {}_nC_r p^r q^{n-r}$.

Example 4. What is the probability of throwing exactly 3 fours in 5 throws of a single die?

Solution.

$$ {}_5C_3 p^3 q^2 = \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 = \frac{125}{3{,}888}. $$

One should not confuse the probability that *exactly* r events will happen in n trials with the probability that at *least* r events or at *most* r events will happen in n trials. The probability that at least r events will happen means that r or more than r events will happen; and the probability that at most r events will happen means that r or less than r events will happen.

THEOREM 5. *If p is the probability that an event will happen in any single trial, then the probability P that an event will happen at least r times in n trials is*

$$ P = p^n + {}_nC_{n-1} p^{n-1} q + {}_nC_{n-2} p^{n-2} q^2 + \cdots + {}_nC_r p^r q^{n-r}. \quad (19) $$

or

$$ P = p^n + {}_nC_1 p^{n-1} q + {}_nC_2 p^{n-2} q^2 + \cdots + {}_nC_r p^r q^{n-r}. \quad (20) $$

Proof: The probability that the event will happen n times in n trials is p^n. By Theorem 4 the probability that the event will happen exactly $n - 1$ times in n trials is ${}_nC_{n-1} p^{n-1} q = {}_nC_1 p^{n-1} q$. The probability that the event will happen exactly $n - 2$ times in n trials is ${}_nC_{n-2} p^{n-2} q^2 = {}_nC_2 p^{n-2} q^2$, and so on. Finally the probability that the event will happen exactly r times in n trials is ${}_nC_r p^r q^{n-r}$. Then by Theorem 3 the probability that any one of these events will happen is the sum of the separate probabilities.

Example 5. Find the probability of throwing at least 3 fours in 5 throws of a single die.

Solution. The probability of throwing either 5 fours in 5 throws, or throwing exactly 4 fours in 5 throws, or throwing exactly 3 fours in 5 throws is given by the following sum.

$$P = {}_5C_5\left(\frac{1}{6}\right)^5 + {}_5C_4\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right) + {}_5C_3\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$$
$$= \left(\frac{1}{6}\right)^5 + {}_5C_1\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right) + {}_5C_2\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$$
$$= \frac{1}{7{,}776} + \frac{25}{7{,}776} + \frac{250}{7{,}776}$$
$$= \frac{23}{648}.$$

In the case of probability of life, life and death are not equally likely to happen, nor is the probability the same for different individuals. However, by observing many individuals we can speak of the probability of life of each individual as being representative of the group. To illustrate, from the American Experience Table of Mortality (p. 411) it is found that out of 92,637 men living at the age of 20, there are 89,032 of them still living five years later. Hence we say the probability that a man 20 years of age will live five more years is

$$p = \frac{89{,}032}{92{,}637},$$

or approximately 24 out of 25.

PROBLEMS

1. One ball is drawn at random from a box containing 3 white and 5 black balls. What is the probability that the ball is (a) white? (b) black?

2. A card is drawn from a deck of 52 cards. What is the probability that the card is red?

3. Two cards are drawn from a deck of 52 cards. What is the probability that they are both face cards?

4. Two cards are drawn from a deck of 52 cards. What is the probability that they form a pair?

5. There are 25 names in a box. If 2 names are drawn at random, what is the probability that (a) a given name is drawn? (b) two given names are drawn?

6. If you are standing in a line of 8 persons, what is the probability that you are at the head of the line? What is the probability that you are standing beside your chum?

7. A box contains tickets numbered 1, 2, 3, 4, and 5, respectively. If five tickets are drawn one at a time, what is the probability that they will be drawn in the order 1, 2, 3, 4, 5?

8. A 5-digit number is written with the digits 1, 2, 3, 4, and 5. What is the probability that the number is even? odd?

9. What is the probability of throwing 3 heads (a) in three throws of a single coin? (b) in one throw of three coins?

10. What is the probability of throwing 4 heads (a) in four throws of a single coin? (b) in one throw of four coins?

11. In a single throw of 5 coins, what is the probability of throwing (a) exactly 3 heads? (b) at least 3 heads?

12. A coin is tossed twice. What is the probability that one fall is head and the other fall is tail?

13. If you have 3 lottery tickets out of 100 tickets and one ticket is drawn at random, what is the probability that you hold the winning ticket?

14. Which is the greater, the probability of throwing at least one ace in six throws of a die, or the probability of throwing at least one head in two throws of a coin?

15. What is the probability of throwing at least one 6 in three throws of a single die?

16. What is the probability of throwing just one pair in a single throw of (a) 2 dice? (b) 3 dice?

17. What is the probability of throwing a total of seven with a single throw of two dice?

18. In throwing 2 dice eight times, what is the probability of throwing (a) 7 exactly five times? (b) 7 at least five times?

19. Four balls are drawn one at a time from a box containing 4 white and 8 black balls. What is the probability that they are alternately of different colors?

20. A box contains 3 girl's names and 3 boy's names. If three consecutive drawings of two names each are made, what is the probability that each of the three drawings will contain a boy's name and a girl's name?

21. What is the probability of throwing 4 heads exactly 5 times in 10 throws of six coins?

22. Ten coins are tossed. What is the probability that exactly 5 coins will fall heads up?

23. Four cards are drawn from a deck of 52 cards. What is the probability that two are red and two are black?

24. A building has 5 exits, each equally accessible. If there are 3 persons in the building when a fire alarm is sounded, what is the probability that they will all run for the same exit?

25. If 4 balls are drawn at random from a box containing 3 red, 5 white, and 8 blue balls, what is the probability that: (a) all are white? (b) all are blue? (c) 3 are red and 1 is white? (d) 2 are white and 2 are

blue? (e) 3 are white and 1 is blue? (f) 1 is red, 1 is white, and 2 are blue?

26. A basket contains 6 red balls, 8 white balls, and 4 blue balls. Three balls are drawn at random. What is the probability that: (a) all three are red? (b) all three are white? (c) all three are blue? (d) 2 are red and 1 is white? (e) 2 are red and 1 is blue? (f) 2 are white and 1 is red? (g) 2 are white and 1 is blue? (h) 2 are blue and 1 is white? (i) 2 are blue and 1 is red? (j) 1 is red, 1 is white, and 1 is blue?

27. At a bridge party of 3 tables, the men draw their tallies from one set of six tallies, and the women draw their tallies from a different set of six tallies. What is the probability that each man will have his wife for a partner?

28. A box contains the names of 3 married couples. If three consecutive drawings of two names each are made, what is the probability that each drawing will contain the names of a husband and wife?

29. Three men go rabbit hunting. If one man gets, on the average, 1 rabbit out of every 2, the second man gets 2 rabbits out of every 3, and the third man gets 3 rabbits out of every 5, what is the probability that they will get a rabbit if they all fire simultaneously? What is the probability that they will all hit the rabbit?

30. In a game of chance 3 dice are tossed. Before tossing, a player calls a number from 1 to 6. If any of the three dice turns up the number called, the player wins double his bet. If two of the dice turn up the number called, the player wins treble. If all three dice turn up the number called, the player wins quadruple. If none of the dice turn up the number called, the player loses. What are the odds in favor of the house?

31. You are given an option of drawing a bill from one of two boxes. In the one box there are 9 one-dollar bills and 1 ten-dollar bill, and in the other box there are 5 one-dollar bills and 8 ten-dollar bills. What is the probability that you will draw a ten-dollar bill?

32. If 3 cards are drawn at random from a deck of 52 cards, what is the probability that: (a) all are aces? (b) all are face cards? (c) all are red? (d) all are the same suit? (e) all are of a kind (same value)? (f) all are the same color? (g) they are consecutive cards in the same suit? (h) they are each of a different suit?

33. What is the probability of dealing a perfect hand at bridge (thirteen cards in one suit)?

34. What is the probability that the dealer at bridge deals himself a perfect hand?

35. What is the probability that a man 21 years of age will live to be 30? 40? 50? 60? 65?

36. What is the probability that a man now 21 years of age will die at the age of 30? 40? 50? 60? 65?

37. At the time of the birth of a son, the father is 30 and the mother is 25. What is the probability that the parents will both be living when the boy is 21 years of age? What is the probability that just one of the parents will be living at this time?

38. A husband was 25 and his wife was 20 at the time of their marriage. What is the probability that at least one of them will live to be 70?

39. A boy 24 and a girl 20 marry. What is the probability that they will live to celebrate their golden wedding anniversary? If they are both living 25 years after marriage, what is the probability that they will live to celebrate their golden wedding anniversary?

40. Fifteen years after marriage the couple in Problem 39 has 2 boys, one aged 10 and the other aged 13. What is the probability that the parents will live to see both boys become of age? What is the probability that all four will live to celebrate the parents' golden wedding anniversary?

CHAPTER XVI

Binomial Theorem

92. BINOMIAL THEOREM; POSITIVE INTEGRAL EXPONENTS

The theory of combinations is very useful for finding the coefficients of the terms in the expansions of powers of binomials. Each term of the expansion of

$$(a + x)^n = (a + x)(a + x)(a + x) \cdots (a + x)(a + x)(a + x)(a + x) \qquad (n \text{ factors})$$

is composed of the product of n letters, where one and only one letter is taken from each of the n pairs of parentheses. Therefore, the number of different ways any product such as $a^{n-r}x^r$ can be formed is either the number of ways the r x's can be selected from the n pairs of parentheses or the number of ways the $n - r$ a's can be selected from the n pairs of parentheses. By Article 89, Equation (7), these two numbers are equal and they are ${}_nC_r$ or ${}_nC_{n-r}$. After the r x's have been selected, the $n - r$ a's can be selected in but one way, or if the $n - r$ a's are selected first, then the r x's can be selected in but the one way. Applying this reasoning to each term, we have the following *formula* for the expansion of a binomial, which is known as the **binomial theorem*** (see Equation 10, Article 89):

$$\begin{aligned}(a + x)^n &= a^n + {}_nC_1a^{n-1} + {}_nC_2a^{n-2} + \cdots + {}_nC_ra^{n-r}x^r + \cdots + {}_nC_1ax^{n-1} + x^n \qquad (1)\\ &= a^n + na^{n-1}x + \frac{n(n-1)}{1\cdot 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}x^3 + \cdots + x^n. \qquad (2)\end{aligned}$$

* A proof of the binomial theorem by mathematical induction is given in Article 103, Chapter XX.

Sometimes the symbol $\binom{n}{r}$ is used for the binomial coefficients instead of the symbol ${}_nC_r$.

Example 1. Expand $(a + x)^6$.

Solution.

$$(a+x)^6 = a^6 + {}_6C_1a^5x + {}_6C_2a^4x^2 + {}_6C_3a^3x^3 + {}_6C_2a^2x^4 + {}_6C_1ax^5 + x^6,$$

$$= a^6 + 6a^5x + \frac{6\cdot 5}{1\cdot 2}a^4x^2 + \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3}a^3x^3 + \frac{6\cdot 5}{1\cdot 2}a^2x^4 + 6ax^5 + x^6;$$

simplifying gives

$$(a+x)^6 = a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6.$$

The **binomial theorem** for the expansion of $(a + x)^n$ has the following properties:

1. *The first term is* $\boldsymbol{a^n}$.
2. *The second term is* $\boldsymbol{na^{n-1}x}$.
3. *The number of terms in the expansion is* $(\boldsymbol{n + 1})$.
4. *The exponents of* $\boldsymbol{a}$ *decrease by unity, and the exponents of* $\boldsymbol{x}$ *increase by unity from term to term.*
5. *The sum of the exponents of* $\boldsymbol{a}$ *and* $\boldsymbol{x}$ *in any term is* $\boldsymbol{n}$.
6. *The coefficients of the terms equidistant from the ends of the expansion are equal.*
7. ${}_n\boldsymbol{C}_0$ *must be defined to be* 1.
8. *If in any term the coefficient is multiplied by the exponent of* $\boldsymbol{a}$, *and this product is divided by the exponent of* $\boldsymbol{x}$ *increased by* 1, *the result is the coefficient of the next term.*

To prove this latter statement, consider the two consecutive terms

$${}_nC_ra^{n-r}x^r + {}_nC_{r+1}a^{n-r-1}x^{r+1}$$

in the binomial theorem, and compare the coefficients ${}_nC_r$ and ${}_nC_{r+1}$. Since

$${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r},$$

and

$${}_nC_{r+1} = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)}{1\cdot 2\cdot 3\cdots(r)\cdot(r+1)},$$

it follows that ${}_nC_{r+1}$ can be obtained from ${}_nC_r$ in the following manner:

$${}_nC_{r+1} = {}_nC_r\frac{n-r}{r+1},$$

which completes the proof. To illustrate, in

$$(a + x)^6 = a^6 + 6a^5x + 15a^4x^2 + 20a^3x^3 + 15a^2x^4 + 6ax^5 + x^6,$$

the third coefficient is $\frac{6 \cdot 5}{2} = 15$, the fourth coefficient is $\frac{15 \cdot 4}{3} = 20$, the fifth coefficient is $\frac{20 \cdot 3}{4} = 15$, and so on.

The objection to using this method is that if an error is made in calculating any coefficient, then the coefficients following will be incorrect. Therefore, when this method is used to find the coefficients, then some of the coefficients should be checked either by evaluating the combination symbol for the coefficient, or by observing that the coefficients of the terms equidistant from the ends of the expansion are equal.

For the nth power of a difference, we can either expand $(a - x)^n$ by making x negative in the formula in Equation (1) and expand $[a + (-x)]^n$; or we can observe that every other term contains an even power of x and hence the terms will alternate plus and minus, the odd terms being positive and the even terms being negative. Hence, we may write

$$(a - x)^n = a^n - {}_nC_1a^{n-1}x + {}_nC_2a^{n-2}x^2 - {}_nC_3a^{n-3}x^3 + \cdots \quad (3)$$

In the expansion of a binomial containing numbers and powers of letters it is best to first write the expansion in formula form, then simplify. This is equivalent to substituting the proper values for a, x, and n in Formula (1), and then simplifying the result. To illustrate:

Example 2. Expand $\left(2x^2 + \frac{a}{2b^3}\right)^4$.

Solution. In this case $n = 4$, $a = 2x^2$, and $x = \frac{a}{2b^3}$. Substituting in Equation (1), and simplifying we have,

$$\left(2x^2 + \frac{a}{2b^3}\right)^4 = (2x^2)^4 + {}_4C_1(2x^2)^3\left(\frac{a}{2b^3}\right) + {}_4C_2(2x^2)^2\left(\frac{a}{2b^3}\right)^2 + {}_4C_3(2x^2)\left(\frac{a}{2b^3}\right)^3 + {}_4C_4\left(\frac{a}{2b^3}\right)^4$$

$$= 16x^8 + 4\,\frac{8x^6a}{2b^3} + \frac{4 \cdot 3}{1 \cdot 2} \cdot \frac{4x^4a^2}{4b^6} + \frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3} \cdot \frac{2x^2a^3}{8b^9} + \frac{4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{a^4}{16b^{12}}$$

$$= 16x^8 + 16x^6\,\frac{a}{b^3} + 6x^4\,\frac{a^2}{b^6} + x^2\,\frac{a^3}{b^9} + \frac{1}{16} \cdot \frac{a^4}{b^{12}}.$$

93. THE GENERAL TERM

Suppose it is required to write any particular term of the binomial expansion. We shall call the general term of the expansion of $(a + x)^n$ the rth term.

Since x does not enter the expansion until the second term, the exponent of x in any term must be *one less* than the number of the term. Therefore, the exponent of x in the rth term is $r - 1$.

Since each term must be of degree n in the two letters a and x, the exponent of a in the rth term must be $n - (r - 1)$ or $n - r + 1$.

From the discussion given in Article 92, the coefficient of the rth term is ${}_nC_{r-1}$ or ${}_nC_{n-r+1}$. Therefore, the rth term in the expansion of $(a + x)^n$ is

$$ {}_nC_{r-1}a^{n-r+1}x^{r-1} \quad \text{or} \quad {}_nC_{n-r+1}a^{n-r+1}x^{r-1}. \tag{4} $$

It is always possible to use the smaller of the two numbers $r - 1$ or $n - r + 1$ when finding the value of ${}_nC_{r-1}$.

Example. Find the eleventh term of the expansion of $(2x - y)^{15}$.

Solution. The rth term is

$$ {}_nC_{r-1}a^{n-r+1}x^{r-1} \quad \text{or} \quad {}_nC_{n-r+1}a^{n-r+1}x^{r-1}. $$

Substituting, we find the 11th term to be

$$ {}_{15}C_{10}(2x)^5(-y)^{10} \quad \text{or} \quad {}_{15}C_5(2x)^5(-y)^{10}; $$

hence, we obtain

$$ \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(32x^5)(y^{10}); $$

simplifying gives

$$ 96{,}096x^5y^{10} $$

for the 11th term.

94. BINOMIAL THEOREM; NEGATIVE AND FRACTIONAL EXPONENTS

The formula given by Equation (1) for the expansion of a binomial was derived on the assumption that n was a positive integer. If the number x lies between $-a$ and $+a$, this theorem can be extended to hold for negative and fractional values of n, although the proof is beyond the scope of this book. If n is not a positive integer, then none of the coefficients in the expansion of $(a + x)^n$ become zero, and therefore there is no last term. These expansions are called **binomial series.**

For negative or fractional values of n, the symbol ${}_nC_r$ becomes meaningless as far as combinations are concerned. However, the coefficients in the binomial series have the values obtained by substituting the given value of n in this symbol. To illustrate:

Since

$$ {}_nC_4 = \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}, $$

we define ${}_{\frac{1}{2}}C_4$ to be

$$ \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{1\cdot 2\cdot 3\cdot 4} = \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1\cdot 2\cdot 3\cdot 4} = -\frac{5}{128}, $$

and ${}_{-2}C_4$ to be

$$ \frac{(-2)(-2-1)(-2-2)(-2-3)}{1\cdot 2\cdot 3\cdot 4} = \frac{(-2)(-3)(-4)(-5)}{1\cdot 2\cdot 3\cdot 4} = 5. $$

Example 1. Find the first five terms in the expansion of $(1+x)^{-1}$.

Solution.

$$ (1+x)^{-1} = 1 + {}_{-1}C_1x + {}_{-1}C_2x^2 + {}_{-1}C_3x^3 + {}_{-1}C_4x^4 + \cdots, $$

$$ = 1 + \frac{-1}{1}x + \frac{-1(-1-1)}{1\cdot 2}x^2 + \frac{-1(-1-1)(-1-2)}{1\cdot 2\cdot 3}x^3 + \cdots, $$

$$ = 1 - x + x^2 - x^3 + x^4 - \cdots. \qquad (|x| < 1) $$

In this case $|x|$ must be less than 1, otherwise the expansion would be meaningless.

Example 2. Find the first five terms in the expansion of $\sqrt{a^2-x^2}$.

Solution.

$$ \sqrt{a^2-x^2} = (a^2-x^2)^{\frac{1}{2}} $$

$$ = (a^2)^{\frac{1}{2}} - \tfrac{1}{2}(a^2)^{-\frac{1}{2}}(x^2) + \frac{(\frac{1}{2})(-\frac{1}{2})}{1\cdot 2}(a^2)^{-\frac{3}{2}}(x^2)^2 - \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{1\cdot 2\cdot 3}(a^2)^{-\frac{5}{2}}(x^2)^3 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{1\cdot 2\cdot 3\cdot 4}(a^2)^{-\frac{7}{2}}(x^2)^4; $$

simplifying gives

$$ \sqrt{a^2-x^2} = a - \frac{1x^2}{2a} - \frac{1x^4}{8a^3} - \frac{1x^6}{16a^5} - \frac{5x^8}{128a^7} \cdots. \qquad (a > x) $$

The following arrangement of numbers is known as Pascal's Triangle and was used for many centuries to make binomial

expansions. The numbers are the coefficients in the expansions for the indicated values of n.

1	$n = 0.$
1 1	$n = 1.$
1 2 1	$n = 2.$
1 3 3 1	$n = 3.$
1 4 6 4 1	$n = 4.$
1 5 10 10 5 1	$n = 5.$

The binomial theorem for $n = 2$ and $n = 3$ was known to the Greeks in the third century before Christ. In 1303, the Chinese algebraist Chu Shi-kie knew the triangular arrangement of the binomial coefficients, which first appeared in print in 1527 in the arithmetic of Apianus. In about 1654, this triangle was investigated in Europe by the French mathematician B. Pascal (1623–1662), who made numerous discoveries relating to this array, and today this array is known as Pascal's Triangle. However, his results were not published until 1665. During this period the mathematicians could write the binomial expansion for any integral power, provided they had the expansion for the next power smaller. It was the Swiss mathematician James Bernoulli (1654–1705) who first gave a proof of the binomial theorem for any positive integral exponent. This was published in 1713, eight years after his death. Euler, in 1773, was the first to give a complete proof of the binomial theorem for all real exponents, although Newton, in 1676, stated the same conclusions but gave no real demonstration of the fact. Abel, in 1826, gave a complete proof of the binomial theorem for real or imaginary exponents.

QUESTIONS

1. What is the formula for the expansion of $(a + x)^n$?
2. How many terms are there in the expansion of $(a + x)^n$?
3. What kind of values can n have in the expansion of $(a + x)^n$?
4. What is the degree in a and x of any term in the expansion of $(a + x)^n$?
5. What is the degree of x in the rth term in the expansion of $(a + x)^n$?
6. What is the degree of a in the rth term in the expansion of $(a + x)^n$?
7. What is the coefficient of the rth term in the expansion of $(a + x)^n$?
8. What is the formula for the rth term in the expansion of $(a + x)^n$?
9. What other term has the same coefficient as the rth term of $(a + x)^n$?

EXERCISES

Write the expansion of each of the expressions in Exercises 1 through 24 and simplify each term of the result.

1. $(a + x)^4$.
2. $(a - x)^5$.
3. $(x + 2)^7$.
4. $(1 - x)^8$.
5. $(x^2 - y^2)^6$.
6. $(xy + 1)^9$.
7. $(a^2 - b)^5$.
8. $(10 + 1)^6$.
9. $(20 - 3)^5$.
10. $\left(3x^2 - \frac{2}{x}\right)^6$.
11. $\left(\frac{a}{b} + \frac{a^2}{b^2}\right)^8$.
12. $\left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)^5$.
13. $\left(\frac{x^2}{2y} + \frac{2y^2}{x^2}\right)^7$.
14. $(a^{\frac{2}{3}} + b^{\frac{3}{2}})^5$.
15. $(\sqrt{x} - \sqrt{y})^6$.
16. $\left(4a^2 - \frac{b}{2}\right)^4$.
17. $\left(\frac{2x}{y^2} - \frac{y^2}{2x}\right)^6$.
18. $(a + x + y)^3$.
19. $(x^{\frac{1}{2}} - x^{-\frac{1}{3}})^6$.
20. $(xy^{-1} + x^{-1}y)^6$.
21. $\left(\frac{2}{3} - x^{-1}\right)^5$.
22. $\left(\frac{1}{2}x - 2x^{-1}\right)^7$.
23. $(x^{-1} + y^{-2})^6$.
24. $\left(x^{\frac{3}{2}} - \frac{1}{2}y^{\frac{2}{3}}\right)^5$.

Find the first *five* terms in the expansion of the expressions in Exercises 25 through 42 and simplify each term.

25. $(x + a)^{20}$.
26. $(a^2 + x)^{30}$.
27. $\left(\frac{x}{y} - \frac{y}{x}\right)^{12}$.
28. $(x - 2^{-2})^{10}$
29. $(x + a)^{2n}$.
30. $(x + a)^{n+2}$.
31. $(1 + x)^{\frac{1}{2}}$.
32. $\sqrt[3]{1 + x}$.
33. $\sqrt{x^2 - \frac{1}{2}}$.
34. $\sqrt[5]{x - \frac{1}{2}}$.
35. $(x + a^{\frac{1}{2}})^{-2}$.
36. $(a + 2x^{-\frac{1}{2}})^{-2}$.
37. $\frac{1}{1 + x}$.
38. $\frac{1}{(x - a)^3}$.
39. $(x - a)^{\frac{2}{3}}$.
40. $(\sqrt[3]{x} + \sqrt[3]{a})^{\frac{2}{3}}$.
41. $(x^2 - a^2)^{-\frac{1}{2}}$.
42. $(x - 2y)^{-\frac{3}{4}}$.

Find the indicated term in each of the following expansions (Exercises 43 through 58):

43. 5th term of $(x + 2y)^{12}$.
44. 16th term of $(xy - 1)^{20}$.
45. 45th term of $(a + x)^{50}$.
46. 33rd term of $(2x^2 + y)^{36}$.
47. 14th term of $(\sqrt{a} + \sqrt{b})^{16}$.
48. 60th term of $(x - \sqrt{y})^{60}$.
49. 8th term of $\left(\frac{2}{3} + \frac{3x}{4}\right)^{10}$.
50. 6th term of $\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{25}$.
51. 7th term of $\sqrt{1 + x}$.
52. 5th term of $\sqrt[3]{x^3 - a^3}$.

53. 12th term of $\dfrac{1}{x^2 - a^2}$.

54. 4th term of $(x^{-2} + x)^{-\frac{1}{2}}$.

55. 4th term of $(x^3 + 2a^3)^{\frac{4}{3}}$.

56. 28th term of $(2x^{-2} - \sqrt{y})^{30}$.

57. Middle term of $\left(x^2 - \dfrac{a^2}{2}\right)^{10}$.

58. Middle term of $(\sqrt{x} - y^2)^8$.

In the expansion of $(a + x)^n$ write the following terms:

59. $(r + 1)$st term.

60. $(r + 5)$th term.

61. $(r - 2)$nd term.

62. $(n - 4)$th term.

Use the binomial theorem to find the value of each of the following:

63. $(21)^4$.

64. $(19)^3$.

65. $(98)^4$.

66. $(29)^5$.

Evaluate the following expressions correct to two decimal places by expanding as $\sqrt{a^2 \pm x}$, $\sqrt[3]{a^3 \pm x}$, or $\sqrt[4]{a^4 \pm x}$:

67. $\sqrt{102}$.

68. $\sqrt{85}$.

69. $\sqrt[3]{124}$.

70. $\sqrt[3]{215}$.

71. $\sqrt[4]{17}$.

72. $\sqrt[4]{14}$.

CHAPTER XVII

Determinants

95. GENERAL DISCUSSION

Early in the seventeenth century the Chinese employed some of the ideas of determinants in their study of systems of linear equations. Leibniz, in 1693, seems to have been the first mathematician in Europe to give a systematic arrangement of what we call the expansion of the determinant of orders two and three. For this reason Leibniz is sometimes called the discoverer of determinants. In 1771, the French mathematician, C. A. Vandermonde (1735–1796) applied these results to solving systems of linear equations and introduced the symbol

$$\begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{matrix}$$

for a determinant of order three. In 1812, A. L. Cauchy developed several general theorems concerning determinants, arranged the elements in a square array without using the vertical lines, and introduced the term determinant. The term determinant was first used by Gauss in 1801, but in a little different sense than we use it today. It was the English mathematician A. Cayley (1821–1895) who, in 1841, first gave us our modern determinant notation.

If we wish to generalize the ideas used in Chapter VIII for determinants of orders two and three, it is necessary to define a determinant independent of its order. Let us express the general determinant with notation similar to the notation used in the case of determinants of orders two and three; that is, let us express the general determinant of order n as a square array of n^2 elements using letters with subscripts so that each column (vertical line) contains the same letter and each row (horizontal line) contains the same subscript, and no two columns contain the same letter

and no two rows the same subscript. Thus we write the following square array

$$\begin{vmatrix} a_1 & b_1 & c_1 & \cdots & r_1 \\ a_2 & b_2 & c_2 & \cdots & r_2 \\ a_3 & b_3 & c_3 & \cdots & r_3 \\ a_n & b_n & c_n & \cdots & r_n \end{vmatrix}$$

to represent a determinant of order n, and lay down the following definition.

DEFINITION. *A determinant of order* n *is an algebraic function of* n^2 *elements called numbers and is written as a square array in these* n^2 *elements. The value of a determinant of order* n *is the algebraic sum of all possible products, consisting of* n *elements each, that can be formed by taking one and only one element from each row and each column. Each product is prefixed with a plus sign if the number of inversions (defined in the following paragraph) of subscripts is even or zero, or a minus sign if the number of inversions of subscripts is odd, when the letters in this product are arranged in the same order as the letters in the first row. (The sign of each product would be determined in the same manner if we used the number of inversions of letters when the subscripts in the product are arranged in the same order as the subscripts in a column.) The indicated sum is called the* **expansion** *of the determinant.*

If, in an arrangement of positive integers, a greater precedes a less, there is said to be an **inversion.** When counting the number of inversions of subscripts it is necessary to compare each subscript with all the following subscripts. To illustrate, in the product $a_2b_4c_1d_3$, 2 before 1 is one inversion of subscripts; likewise, 4 before 1, and 4 before 3. Hence, there are three inversions of subscripts when the letters in the product are arranged in order, and therefore this product must be prefixed with a minus sign and we write

$$-a_2b_4c_1d_3.$$

Had we arranged the subscripts in order we would have $c_1a_2d_3b_4$, and the inversions of letters are c before a, c before b, and d before b. Thus, in either case there are three inversions, and the sign of this product must be changed.

To illustrate, we shall expand the following determinant of order three.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3. \quad (1)$$

The student should determine the number of inversions of subscripts in each product and thus determine the respective signs of the products. He should also observe that each product contains all three letters and all three subscripts, and that this expansion contains all such products that can be formed in this manner.

Let us determine the number of products that can be formed in the expansion of a determinant of order n. Since the n elements composing a product are taken one from each row and one from each column, it follows that the element from the first column can be chosen in n ways; the element from the second column can be chosen in $(n - 1)$ ways; the element from the third column can be chosen in $(n - 2)$ ways; and so on. Hence, the total number of products that can be formed is

$$n(n-1)(n-2)(n-3) \cdots 4 \cdot 3 \cdot 2 \cdot 1 = n! \quad (2)$$

Thus the expansion of a determinant of order two has 2 products, a determinant of order three has $3 \cdot 2 \cdot 1 = 6$ products, a determinant of order four has $4 \cdot 3 \cdot 2 \cdot 1 = 24$ products, and so forth. For determinants of very large order, it would be a very long and difficult task to find their values by first determining all the possible products. Hence, certain properties have been developed to help simplify the work of finding the value of a determinant. We shall give only a few of these properties and shall illustrate each property by special cases. We shall give only an inductive proof of the general cases. However, it should be observed that the generality of the proofs is not lost, because the proofs are independent of the order of the determinant.

THEOREM I. *If in a determinant the columns, when read from left to right, are interchanged with corresponding rows, when read from top to bottom, the value of the determinant is unchanged.* To illustrate:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \quad (3)$$

In the resulting determinant the subscripts denote columns and the letters rows. If any product in the expansion of the resulting determinant is arranged with subscripts in order, the number of inversions of the letters is the same as the number of inversions of the subscripts when the letters in the product are arranged in order. Thus, the product

$$a_2b_3c_1 \qquad \text{and} \qquad c_1a_2b_3,$$

which is common to the expansion of the two determinants, has either two inversions of subscripts or two inversions of letters, and hence will have the same sign in the expansion of either determinant.

THEOREM II. *If two rows, or two columns, of a determinant are interchanged, the sign of the determinant is changed, but its numerical value is not changed.* To illustrate:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \qquad \text{or} \qquad -\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}. \tag{4}$$

Suppose the letters and subscripts in the given determinant are in order. If we interchange any two columns in the determinant, the order of the subscripts in the columns will not be changed. Let us interchange two columns, supposing there are k columns between the two columns interchanged. Let us now compare the order of the letters in the two determinants. In the new determinant the letter in the column moved to the left will now precede the k letters, thus giving k inversions of the letters; these k letters also precede the letter in the column moved to the right, thus giving k more inversions; there is 1 inversion of the letters in the two columns interchanged. Therefore, each product in the new determinant has $2k + 1$ more inversions of letters than this same product has in the original determinant. But $2k + 1$ is an odd number, and an odd number of interchanges of adjacent rows changes the sign of the determinant. Thus the like products in the expansions of the two determinants will have the same numerical value but any given product must be prefixed by a plus sign in the expansion of the one determinant and by a minus sign in the expansion of the other determinant. Hence the two determinants will have the same numerical value but will be opposite in sign.

THEOREM III. *If a determinant has two rows or two columns that are identical, the value of the determinant is zero.* To illustrate:

$$\begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix} = 0. \tag{5}$$

Interchanging the two identical rows or columns will not alter the form of the determinant, and hence will not alter its value. However, by Theorem II, when two rows or two columns are interchanged the sign of the determinant is changed. Since zero is the only number whose value is not changed by changing its sign, the value of the determinant must be 0.

THEOREM IV. *If all the elements of any row or any column have a common factor, this common factor may be taken out and written as a coefficient of the resulting determinant.* To illustrate:

$$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \tag{6}$$

Since each product in the expansion of the given determinant contains an element from each row and each column, it follows that each of these products must contain the common factor. After factoring out this common factor, the sum of the other factors of these products is the expansion of a determinant like the given determinant except for the row or column containing the common factor, and this row or column has the common factor removed.

THEOREM V. *If each element of a certain column or each element of a certain row is composed of the sum of two numbers, the determinant can be expressed as the sum of two determinants in the manner indicated below.*

$$\begin{vmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{vmatrix}. \tag{7}$$

Each product in the expansion of the first determinant is of the form $(a + x)bc$, and upon removing parentheses, this product can be written as the sum of two terms like $abc + xbc$. Therefore the sum of all the products of the form abc constitute the one determinant and the sum of all the products of the form xbc constitute the second determinant.

Most of the preceding properties have been developed primarily to aid in proving the following theorem which is very useful when finding the value of determinants of any order greater than two.

THEOREM VI. *The value of any determinant is unchanged if any column in the given determinant is replaced by another column in which the elements are formed by adding to the elements in the original column the product of* **m** *times the corresponding elements in another column.* (*The same thing applies to rows.*) To illustrate:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 + ma_1 \\ a_2 & b_2 & c_2 + ma_2 \\ a_3 & b_3 & c_3 + ma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m \begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \tag{8}$$

If the elements of the first column are multiplied by m and added to the elements of the third column, then the resulting determinant by Theorem V is equal to the sum of two determinants. One of these determinants is identical with the original determinant, and the other determinant, after removing the common factor m, has its first and third columns equal. By Theorem III the second determinant is equal to zero.

Observe that the column (or row) whose elements are multiplied by m must be present in the new determinant, and that it is the column (or row) to which these products are added that is replaced by the sum.

96. DEVELOPMENT BY MINORS

Let us next consider all the terms in the expansion of a determinant of order n that contain a given element. It should be observed that these terms can be written as a determinant of order $n - 1$ with this common element as a coefficient. This determinant of order $n - 1$ is called the minor with respect to this common element; that is, *the* **minor** *with respect to any element is the determinant obtained from the original determinant by omitting the column and row containing this common element.* To illustrate, in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

the **minor** with respect to the element b_1 is

$$\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}.$$

The minors with respect to the elements a_1, a_2 and a_3 are

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

In accordance with Equation (1), the expansion of the foregoing determinant of order three may be written as

$$\begin{aligned} &a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2 \\ &\quad = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1). \end{aligned}$$

In this latter form it is not difficult to see that the quantities in the the parentheses are the values of the minors with respect to the elements a_1, a_2, and a_3. Therefore we may write

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \tag{9}$$

which is the expansion of this determinant by minors with respect to the elements in the first column.

A determinant of any order can be expanded by minors according to the elements of any column or any row. We shall state without proof the following theorem.

THEOREM VII. *A determinant of order* **n** *may be expressed as the sum of the* **n** *products formed by multiplying each element of any column (or row) by its minor, where the sign preceding the product of a given element and its minor is plus or minus according to whether the sum of the number of the column and the number of the row containing this element is even or odd.*

The sign of the product of any element and its minor is determined by the number of interchanges of columns and interchanges of rows necessary to move this element to the upper left-hand corner, keeping the remaining letters and subscripts in order. The sign determined in this manner for the product of any element and its minor is either plus or minus according to whether the sum of the number of the column and the number of the row containing this element is an even or an odd number.

If we expand the following determinant by minors with respect to the elements of the second row, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \tag{10}$$

$$= -a_2(b_1c_3 - b_3c_1) + b_2(a_1c_3 - a_3c_1) - c_2(a_1b_3 - a_3b_1).$$

The sign of the product of c_2 and its minor, for example, is negative because c_2 is located in the 3rd column and the 2nd row and the sum $3 + 2 = 5$ is an odd number.

A second method for determining the sign of the product of any element and its minor is to begin in the upper left-hand corner with plus and then alternate signs, passing from element to element in a vertical or horizontal manner until reaching the given element. Thus, to determine the sign that precedes the product of c_2 and its minor, we may proceed as follows: $+a_1$, $-b_1$, $+b_2$, $-c_2$. Hence the sign is minus. The result will be the same regardless of the order followed, provided we always move horizontally or vertically. Thus we may proceed as follows: $+a_1$, $-a_2$, $+a_3$, $-b_3$, $+c_3$, $-c_2$.

Example 1. Find the value of the determinant

$$\begin{vmatrix} 1 & -2 & 3 \\ 3 & 4 & -2 \\ 2 & 5 & 2 \end{vmatrix}.$$

Solution. Let us expand by minors with respect to the elements in the first column; then

$$\begin{vmatrix} 1 & -2 & 3 \\ 3 & 4 & -2 \\ 2 & 5 & 2 \end{vmatrix} = 1\begin{vmatrix} 4 & -2 \\ 5 & 2 \end{vmatrix} - 3\begin{vmatrix} -2 & 3 \\ 5 & 2 \end{vmatrix} + 2\begin{vmatrix} -2 & 3 \\ 4 & -2 \end{vmatrix}$$

$$= 1(8 + 10) - 3(-4 - 15) + 2(4 - 12)$$

$$= 18 + 57 - 16$$

$$= 59.$$

Expanding determinants of large order by minors may be a long process unless several of the elements in some row or column are zero. By employing Theorem VI it is always possible to find an equivalent determinant with $(n - 1)$ zeros in a given row or in a given column. Then upon expanding this equivalent determinant by minors with respect to the elements in the row or column containing the zeros, we see that the coefficients of all the minors, except one, are zero. Hence, we need only to find the value of one minor, the minor whose coefficient is not zero. Thus, the

work for finding the value of a determinant or order n can be reduced to that of finding the value of a determinant of order $(n-1)$. To illustrate:

Example 2. Find the value of the determinant

$$\begin{vmatrix} 1 & -2 & 3 \\ 3 & 4 & -2 \\ 2 & 5 & 2 \end{vmatrix}.$$

Solution. Let us obtain an equivalent determinant in which the last two elements in the first row are zero. To accomplish this, multiply the elements of the first column by 2 and add the products to the elements of the second column; then multiply the elements of the first column by 3 and subtract the products from the elements in the third column. Thus,

$$\begin{vmatrix} 1 & -2 & 3 \\ 3 & 4 & -2 \\ 2 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 3 & 10 & -11 \\ 2 & 9 & -4 \end{vmatrix} = 1\begin{vmatrix} 10 & -11 \\ 9 & -4 \end{vmatrix} = 59.$$

Observe that the elements in the first column which were multiplied by 2 and 3, respectively, are the elements of the first column in the second determinant.

Example 3. Find the value of the determinant

$$\begin{vmatrix} 2 & -1 & 3 & 4 \\ -3 & 2 & -1 & -2 \\ 1 & 0 & 2 & 3 \\ 3 & -2 & 0 & -1 \end{vmatrix}.$$

Solution.

$$\begin{vmatrix} 2 & -1 & 3 & 4 \\ -3 & 2 & -1 & -2 \\ 1 & 0 & 2 & 3 \\ 3 & -2 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 & 4 \\ 1 & 0 & 5 & 6 \\ 1 & 0 & 2 & 3 \\ -1 & 0 & -6 & -9 \end{vmatrix}$$

$$= -(-1)\begin{vmatrix} 1 & 5 & 6 \\ 1 & 2 & 3 \\ -1 & -6 & -9 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -1 & -3 \\ 0 & -4 & -6 \\ -1 & -6 & -9 \end{vmatrix}$$

$$= +(-1)\begin{vmatrix} -1 & -3 \\ -4 & -6 \end{vmatrix} = 6.$$

Since the second column contains a 1 (plus or minus) it is easy to obtain a new determinant with three zeros in the second column.

For the first row in the new determinant, use the first row in the given determinant. To obtain the second row with a zero for the second element, multiply the elements in the first row by 2 and add the products to the corresponding elements in the second row. The third row has a zero for the second element. To obtain the fourth row with a zero for the second element, multiply the elements in the first row by -2 and add the products to the corresponding elements in the fourth row.

If we expand this fourth order determinant by minors with respect to the elements in the second column, we need only to find the value of the minor with respect to the element -1, as the other minors have zero for their coefficients.

To find the value of the resulting determinant of order three, add the elements in the third row to the elements of the first and second rows, respectively, as is illustrated.

A third method for finding the value of a determinant of any order consists of writing a new determinant with the proper coefficient, the order of the new determinant being one less than the order of the original determinant and the elements of the new determinant being the values of certain determinants of order two. To illustrate:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \frac{1}{a_1} \begin{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} \end{vmatrix}. \tag{11}$$

To prove this, multiply the elements of the second and third rows by a_1 and divide the resulting determinant by a_1^2. Then in the resulting determinant, we can get two zeros in the first column by multiplying the elements in the first row by a_2 and subtracting the products from the elements in the second row, and multiplying the elements in the first row by a_3 and subtracting the products from the elements in the third row. Thus, we have

$$\begin{aligned} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= \frac{1}{a_1^2} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1a_2 & a_1b_2 & a_1c_2 \\ a_1a_3 & a_1b_3 & a_1c_3 \end{vmatrix} \\ &= \frac{1}{a_1^2} \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & a_1b_2 - a_2b_1 & a_1c_2 - a_2c_1 \\ 0 & a_1b_3 - a_3b_1 & a_1c_3 - a_3c_1 \end{vmatrix}. \end{aligned} \tag{12}$$

This result may be written as Equation (11) above.

This process may be extended to determinants of higher order. For a determinant of order four, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \frac{1}{{a_1}^2} \begin{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & d_1 \\ a_3 & d_3 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_4 & b_4 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_4 & c_4 \end{vmatrix} & \begin{vmatrix} a_1 & d_1 \\ a_4 & d_4 \end{vmatrix} \end{vmatrix}. \qquad (13)$$

Example 4. Find the value of

$$\begin{vmatrix} 2 & -1 & 3 & 4 \\ -3 & 2 & -1 & -2 \\ 1 & 0 & 2 & 3 \\ 3 & -2 & 0 & -1 \end{vmatrix}.$$

Solution.

$$\begin{vmatrix} 2 & -1 & 3 & 4 \\ -3 & 2 & -1 & -2 \\ 1 & 0 & 2 & 3 \\ 3 & -2 & 0 & -1 \end{vmatrix} = \frac{1}{2^2} \begin{vmatrix} 1 & 7 & 8 \\ 1 & 1 & 2 \\ -1 & -9 & -14 \end{vmatrix}$$

$$= \frac{1}{4} \cdot \frac{1}{1} \begin{vmatrix} -6 & -6 \\ -2 & -6 \end{vmatrix} = 6.$$

If this process is extended to determinants of order 5, we must multiply the new determinant by $\frac{1}{{a_1}^3}$. For determinants of order six, we must multiply the new determinant by $\frac{1}{{a_1}^4}$, and so on, remembering that a_1 stands for the element in the upper left-hand corner. If the given determinant is of order n, the new determinant must be multiplied by $\frac{1}{{a_1}^{n-2}}$.

This method cannot be used when $a_1 = 0$. If $a_1 = 0$, it is necessary to interchange some rows or columns so that the upper left-hand element is different from zero.

If a system of n linear equations in n unknowns has a solution, the value of each unknown can be written as the ratio of two determinants in a manner similar to the case of linear equations in two and three unknowns. This method of solution is frequently referred to as Cramer's rule, after Gabriel Cramer (1704–1752).

a Swiss mathematician. *In this solution the value of any unknown is written as the ratio of two determinants. The common determinant in the denominator is called the determinant of the system of equations and is formed from the coefficients of the* **n** *unknowns; the first column of this determinant consists of the coefficients of one of the unknowns; the second column consists of the coefficients of a second unknown, and so on. The determinant in the numerator is like the determinant in the denominator except for one column, the column containing the coefficients of the unknown whose value is being sought; this column is replaced by the respective constants which are written on the right-hand side of the equality sign.* The given system has a solution if the value of the determinant in the denominator is different from zero.

QUESTIONS

1. Give six theorems pertaining to determinants.

2. Give three methods for finding the value of a determinant.

3. What is the formula for the value of any unknown in the solution of a system of n linear equations in n unknowns?

EXERCISES

Find the value of each of the following determinants (Exercises 1 through 20):

1. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}$.

2. $\begin{vmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \\ 9 & 8 & 7 \end{vmatrix}$.

3. $\begin{vmatrix} 2 & -3 & 4 \\ 3 & 1 & -6 \\ -5 & 2 & 3 \end{vmatrix}$.

4. $\begin{vmatrix} -3 & 2 & 4 \\ 0 & 7 & -2 \\ 5 & 3 & 5 \end{vmatrix}$.

5. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

6. $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$.

7. $\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix}$.

8. $\begin{vmatrix} -1 & 1 & -1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & -1 & 0 & -1 \\ 2 & 0 & -2 & 1 \end{vmatrix}$.

9. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}$.

10. $\begin{vmatrix} 2 & 1 & 0 & -3 \\ -1 & 2 & 3 & -2 \\ 0 & 1 & 2 & 3 \\ -3 & 2 & 5 & -4 \end{vmatrix}$.

11. $\begin{vmatrix} 1 & -2 & 3 & 0 \\ 2 & -1 & 0 & 2 \\ -1 & 0 & 1 & 3 \\ 0 & 2 & 3 & -1 \end{vmatrix}$.

12. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{vmatrix}$.

13. $\begin{vmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 \end{vmatrix}$.

14. $\begin{vmatrix} 3 & 2 & 1 & 3 \\ 4 & 2 & 3 & 2 \\ 2 & 1 & 3 & 2 \\ 2 & 1 & 3 & 2 \end{vmatrix}$.

15. $\begin{vmatrix} 2 & -1 & 3 & -2 \\ 1 & 3 & 1 & -1 \\ 3 & -2 & 2 & 3 \\ 2 & 1 & -3 & 5 \end{vmatrix}$.

16. $\begin{vmatrix} 2 & 3 & 2 & 3 \\ 3 & 1 & 3 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & 4 & 3 \end{vmatrix}$.

17. $\begin{vmatrix} 1 & a & a & a \\ 1 & b & a & a \\ 1 & a & b & a \\ 1 & a & a & b \end{vmatrix}$.

18. $\begin{vmatrix} 2 & -3 & 4 & -5 \\ 3 & -2 & 5 & -4 \\ -4 & 5 & -2 & 3 \\ 5 & 2 & -3 & 4 \end{vmatrix}$.

19. $\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{vmatrix}$.

20. $\begin{vmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{vmatrix}$.

21. Give two pairs of values of x and y that will satisfy each of the following equations:

(a) $\begin{vmatrix} x & y & 1 \\ 2 & -3 & 1 \\ -4 & 5 & 1 \end{vmatrix} = 0$;

(b) $\begin{vmatrix} x & y & 1 \\ -1 & 3 & 2 \\ 2 & -4 & 3 \end{vmatrix} = 0$.

22. What should be the sign of each of the following if they represent some product from determinants written with letters and subscripts in order?

(a) $a_1b_5c_2d_4e_3$;

(b) $b_2a_4c_1d_3$;

(c) $c_1d_2a_3e_4f_5b_6$;

(d) $c_2e_4a_1b_3d_5$.

23. Solve each of the following equations:

(a) $\begin{vmatrix} x & 1 & 2 \\ -1 & 2 & 3 \\ 3 & 2 & 4 \end{vmatrix} = 0$;

(b) $\begin{vmatrix} x & 2 & 3 \\ 1 & 2 & 3 \\ -1 & 3 & 4 \end{vmatrix} = 0$;

(c) $\begin{vmatrix} x & 1 & 2 \\ 3 & 1 & 2 \\ 3 & x & 2 \end{vmatrix} = 0$.

24. Solve each of the following equations:

(a) $\begin{vmatrix} 2 & 3 & -4 \\ 1 & x & -2 \\ 3 & -5 & 1 \end{vmatrix} = 0$;

(b) $\begin{vmatrix} x & 6 & 4 \\ 2 & x & 3 \\ 4 & 9 & 6 \end{vmatrix} = 0$;

(c) $\begin{vmatrix} 1 & -4 & x \\ 4 & 5 & 2 \\ 2 & x & 1 \end{vmatrix} = 0$.

Solve each of the following systems of equations (Exercises 25 through 30):

25. $3w - 2x + 2y - z = 2,$
$w + 3x - y - 2z = 6,$
$2w - x + 3y + z = 0,$
$w + x + y + z = -1.$

26. $3x - 2y + 3z + 2w = 11,$
$x - 3y + 2z - 4w = -10,$
$-2x + 2y - 3z - 5w = -22,$
$-x - y + z + 3w = 12.$

27. $2w - x + y - z = 3,$
$w + 2x + 3y + 4z = -3,$
$2w - 3x + 2y + 2z = 0,$
$3w - 2x - y + 2z = 1.$

28. $4x + 2y - w = 7,$
$-3y - 2z + w = -4,$
$2x + 3z + 4w = -4,$
$-3x + 4y + 5z = 2.$

29. $w + x = 3,$
$x + y = 4,$
$y - z = 1,$
$z + w = 2.$

30. $2x - 3y + 4z = -4,$
$2y + 3z + 5u = 7,$
$-3z + 2u + 3v = 14,$
$2u - 3v - 2x = 9,$
$-3v + 2x + 5y = -5.$

31. The general equation of a straight line is $y = mx + b$. (a) Find the equation of a straight line passing through the points (2, 3) and (−3, 4); (b) through the points (−3, 0) and (2, −4); (c) through the points (−3, −2) and (5, −6).

Solution. The coördinates of each point must satisfy the equation $y = mx + b$. Therefore, for part (a) the values of m and b can be found by solving the system of equations $3 = 2m + b$ and $4 = -3m + b$.

32. The general equation of a circle is $x^2 + y^2 + ax + by + c = 0$. (a) Find the equation of the circle passing through the three points (1, 0), (−2, 3), and (0, 2); (b) through the three points (5, 0), (−5, 0), and (3, 4); (c) through the three points (1, 2), (1, 3), and (2, 5).

33. The general equation of a conic with its axis parallel to the coördinate axes is $ax^2 + cy^2 + dx + ey + f = 0$. Find the equation of the conic passing through the four points

(a) (1, 1), (2, −1), (0, −2), and(−1, −1);
(b) (1, 2), (3, 0), (0, −1), and (−1, 1);
(c) (2, −1), (3, 1),(5, 2), and (2, 0);
(d) (−3, −4), (3, 5), (1, 0), and (−1, −3).

Suggestion: For each conic find four equations in the five letters a, c, d, e; and f; then solve for any four of the letters in terms of the remaining letter.

34. The general equation of a conic is $ax^2 + bxy + cy^2 + dx + ey + f = 0$. Find the equation of the conic passing through the five points

(a) (1, 2), (3, 0), (0, −1), (0, 0), and (−1, 1);
(b) (0, −4), (1, 1), (2, 3), (4, 0), and (−2, −2);
(c) (0, 1), (2, 2), (1, 1), (1, −1), and (−2, −1);
(d) (0, 0), (0, 1), (3, 1), (−4), −1), and (1, 2);
(e) (1, 1), (2, −1), (0, −2),(−1, −1), and (−2, 1).

Suggestion: For each conic find five equations in the six letters a, b, c, d, e, and f; then solve for any five of the letters in terms of the remaining letter.

CHAPTER XVIII

Partial Fractions

97. GENERAL DISCUSSION

In Chapter IV we learned how to find the sum of several fractions, and that this sum was expressed as a single fraction whose denominator was the least common multiple of the denominators of the given fractions.

In this chapter we shall show how to perform the inverse operation; that is, to decompose a given fraction into a sum of several fractions called **partial fractions.** To illustrate:

$$\frac{x^6 - 6x^4 - 20x^3 + 129x^2 - 252x + 117}{(2x+3)(x-2)^2(x^2-2x+3)^2} = \frac{1}{2x+3} + \frac{2}{x-2} - \frac{1}{(x-2)^2} - \frac{2x-5}{x^2-2x+3} - \frac{4x-3}{(x^2-2x+3)^2}.$$

In what follows we shall assume that the fraction to be resolved into its partial fractions is a proper fraction. A **proper fraction** is a fraction whose numerator is of lower degree than the denominator.

We shall state without proof that it is possible to decompose any proper fraction whose numerator and denominator are polynomials in x into the sum of partial fractions, where the partial fractions satisfy the following conditions. (If the given fraction has the properties of partial fractions, then it will resolve into itself.)

PROPERTY (1). *The least common denominator of all the partial fractions must be the denominator of the given fraction.*

PROPERTY (2). *Each partial fraction must have but a single real prime factor for its denominator.*

PROPERTY (3). *Each partial fraction must be a proper fraction with numerator just one degree less than the degree of the real prime factor in the denominator.*

PROPERTY (4). *For every linear factor of the form* $(ax + b)$ *that occurs* n *times as a factor in the denominator of the given fraction, there must be provision for* n *partial fractions of the form*

$$\frac{A_1}{ax+b}, \quad \frac{A_2}{(ax+b)^2}, \quad \frac{A_3}{(ax+b)^3}, \quad \cdots, \quad \frac{A_n}{(ax+b)^n}.$$

PROPERTY (5). *For every real prime factor of the form* $(ax^2 + bx + c)$, $(b^2 - 4ac < 0)$, *that occurs* n *times as a factor in the denominator of the given fraction, there must be provision for* n *partial fractions of the form*

$$\frac{A_1 + B_1x}{ax^2+bx+c}, \quad \frac{A_2 + B_2x}{(ax^2+bx+c)^2},$$
$$\frac{A_3 + B_3x}{(ax^2+bx+c)^3}, \quad \cdots, \quad \frac{A_n + B_nx}{(ax^2+bx+c)^n}.$$

To illustrate, if we wish to resolve the foregoing fraction into its partial fractions, then it is necessary to write the partial fractions as shown in the following equation:

$$\frac{x^6 - 6x^4 - 20x^3 + 129x^2 - 252x + 117}{(2x+3)(x-2)^2(x^2-2x+3)^2} = \frac{A}{2x+3} +$$
$$\frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{Dx+E}{x^2-2x+3} + \frac{Fx+G}{(x^2-2x+3)^2}. \qquad (1)$$

PROPERTY (6). *The total number of constants* A, B, C, $\cdots$, *and so forth, appearing in the partial fractions must be the same as the degree of the denominator of the given fraction.*

This is evident from the fact that if the given denominator is of degree n, then the numerator of the sum of the several partial fractions is a polynomial of degree $n - 1$ and therefore should contain n constants. A general polynomial of degree $n - 1$ contains n constants.

In the foregoing example the degree of the denominator is 7, and hence there are 7 constants A, B, C, $\cdots$ in the partial fractions.

Since the denominator of the sum of the several fractions is the same as the denominator of the given fraction, and two fractions having the same denominator can be equal only if the two numerators are equal, it follows that:

PROPERTY (7). *After the partial fractions have been added, the numerator of the sum must be identical with the numerator of the*

given fraction; that is, the two numerators must be equal for all values of x *whether real or imaginary, and the coefficients of the like powers of* x *in the two numerators must be equal.*

To find the values of the n constants in the partial fractions it is necessary to set up as many equations as there are constants. These n equations can be formed either by assigning to x n different values and substituting these n values in the numerator of the given fraction and in the numerator of the sum, or by equating the coefficients of the n different powers of x in the two numerators. A combination of the two methods may also be used to form the n equations.

We shall be interested in decomposing into partial fractions only those fractions whose denominators can be expressed as the product of linear and quadratic prime factors.

If the numerator of the given fraction is of equal or higher degree than the denominator, then the given fraction must be reduced by division to the sum of an integral rational expression and a proper fraction, as

$$\frac{3x^4 + 2x^3 + 8x^2 - 5}{x^3 + 2x - 3} = 3x + 2 + \frac{2x^2 + 5x + 1}{(x + 1)(x^2 + x + 3)},$$

and then the proper fraction must be decomposed into partial fractions.

We shall illustrate the preceding discussion by the following examples.

Example 1. Resolve $\dfrac{2x - 14}{(x - 1)(x - 2)(x + 3)}$ into partial fractions.

Solution. Set

$$\frac{2x - 14}{(x - 1)(x - 2)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x + 3}.$$

Adding the partial fractions gives

$$\frac{A(x - 2)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x - 2)}{(x - 1)(x - 2)(x + 3)}.$$

Therefore,

$$2x - 14 \equiv A(x - 2)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x - 2).$$

When solving for the constants A, B, C, $\cdots$, and so on by assigning values to x in both members of the equation, it simplifies the work if we assign to x, whenever possible, those values which will make some of the

factors on the right-hand side of the equation equal to zero. Thus, in this problem we shall let $x = 1$, 2, and -3, respectively.

Let $x = 1$, then $2 - 14 = A(-1)(4)$, or $A = 3$.
Let $x = 2$, then $4 - 14 = B(1)(5)$, or $B = -2$.
Let $x = -3$, then $-6 - 14 = C(-4)(-5)$, or $C = -1$.

Therefore,

$$\frac{2x - 14}{(x - 1)(x - 2)(x + 3)} = \frac{3}{x - 1} - \frac{2}{x - 2} - \frac{1}{x + 3}.$$

Example 2. Resolve $\dfrac{5x^2 - 4}{x^2(x - 2)}$ into partial fractions.

Solution. Set

$$\frac{5x^2 - 4}{x^2(x - 2)} = \frac{A}{x - 2} + \frac{B}{x^2} + \frac{C}{x}.$$

Adding the partial fractions gives

$$\frac{Ax^2 + B(x - 2) + Cx(x - 2)}{x^2(x - 2)}.$$

Therefore,

$$\begin{aligned} 5x^2 - 4 &\equiv Ax^2 + B(x - 2) + Cx(x - 2) \\ &\equiv (A + C)x^2 + (B - 2C)x - 2B. \end{aligned}$$

In this example there are only two real values of x that will make some of the factors in the right-hand member equal to zero; they are $x = 0$ and $x = 2$. For the third equation we can either assign any value to x, or we can equate the coefficients of some like powers of x. We shall equate the coefficients of x^2.

Let $x = 0$, then $-4 = -2B$, or $B = 2$.
Let $x = 2$, then $20 - 4 = 4A$, or $A = 4$.

If we equate the coefficients of x^2, $5 = A + C$; hence $C = 1$, since $A = 4$.

Therefore,

$$\frac{5x^2 - 4}{x^2(x - 2)} = \frac{4}{x - 2} + \frac{2}{x^2} + \frac{1}{x}.$$

It should be observed, for example, that it is impossible to resolve $\dfrac{5x^2 - 4}{x^2(x - 2)}$ into two partial fractions of the form $\dfrac{A}{x - 2} + \dfrac{B}{x^2}$, for if we set

$$\frac{5x^2 - 4}{x^2(x - 2)} = \frac{A}{x - 2} + \frac{B}{x^2}$$

and add the partial fractions we obtain

$$\frac{Ax^2 + B(x - 2)}{x^2(x - 2)}.$$

Then

$$5x^2 - 4 \equiv Ax^2 + B(x - 2),$$

or

$$5x^2 - 4 \equiv Ax^2 + Bx - 2B,$$

which is impossible, since B, the coefficient of x, must be zero, and at the same time the constant term $-2B$ must be -4, or B must be 2.

This illustrates the necessity for including partial fractions with all successive powers of repeated prime factors as denominators, as is indicated in Properties (4) and (5).

Example 3. Resolve $\dfrac{x^3 - x^2 - 9x + 6}{(x^2 + 2)(x - 1)^2}$ into partial fractions.

Solution. Set

$$\frac{x^3 - x^2 - 9x + 6}{(x^2 + 2)(x - 1)^2} = \frac{Ax + B}{x^2 + 2} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}.$$

Adding the partial fractions gives

$$\frac{(Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 2) + D(x^2 + 2)}{(x^2 + 2)(x - 1)^2}.$$

Therefore,

$$\begin{aligned} x^3 - x^2 - 9x + 6 & \\ &\equiv (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 2) + D(x^2 + 2) \\ &\equiv (A + C)x^3 + (-2A + B - C + D)x^2 + \\ &\qquad (A - 2B + 2C)x + (B - 2C + 2D). \end{aligned}$$

In this example the only values of x that will make some of the factors in the right-hand member equal to zero are $x = 1$ and $x = \pm\sqrt{2}i$.

We shall illustrate how some of the constants can be determined by assigning imaginary values to x. But first, let $x = 1$, then

$$1 - 1 - 9 + 6 = D(1 + 2) \qquad \text{or} \qquad D = -1.$$

Let $x = \sqrt{2}i$, then

$$-2\sqrt{2}i + 2 - 9\sqrt{2}i + 6 = (A\sqrt{2}i + B)(-1 - 2\sqrt{2}i);$$

combining gives

$$8 - 11\sqrt{2}i = (4A - B) + (-A - 2B)\sqrt{2}i;$$

equating reals, we have

$$4A - B = 8;$$

equating imaginaries gives

$$-A - 2B = -11.$$

Solving this system by addition, we have

$$\begin{aligned} 8A - 2B &= 16 \\ A + 2B &= 11 \\ \hline 9A \qquad &= 27 \end{aligned}$$

or $A = 3$. Substituting, we obtain

$$3 + 2B = 11,$$

or $B = 4$. Equating coefficients of x^3 gives

$$A + C = 1;$$

then $C = -2$ since $A = 3$.

Therefore,

$$\frac{x^3 - x^2 - 9x + 6}{(x^2 + 2)(x - 1)^2} = \frac{3x + 4}{x^2 + 2} - \frac{2}{x - 1} - \frac{1}{(x - 1)^2}.$$

If we wished to determine the constants by equating coefficients of like powers of x, it would be necessary to solve the following system of four linear equations in four unknowns.

$$\begin{aligned} \text{Equating the coefficients of } x^3, \quad & A + C = 1, \\ \text{Equating the coefficients of } x^2, \quad & -2A + B - C + D = -1, \\ \text{Equating the coefficients of } x, \quad & A - 2B + 2C = -9, \\ \text{Equating the constant terms,} \quad & B - 2C + 2D = 6. \end{aligned}$$

This system can be solved for A, B, C, and D either by substitution, by addition and subtraction, or by determinants.

EXERCISES

Resolve each of the fractions in Exercises 1 through 44 into its simplest partial fraction. (The results can always be checked by adding the fractions.)

1. $\dfrac{5x - 7}{(x - 1)(x - 2)}$.

2. $\dfrac{x + 5}{(x - 2)(x + 1)}$.

3. $\dfrac{x^2 + 5x + 1}{(x + 2)(x^2 + 1)}$.

4. $\dfrac{7}{6x^2 + x - 2}$.

5. $\dfrac{8x - 18}{6x^2 + 11x - 10}$.

6. $\dfrac{2x - 3}{x^2 + 5x + 6}$.

7. $\dfrac{2x^2 - 5x + 17}{x^3 - x^2 - 8x + 12}$.

8. $\dfrac{x + 2}{x^3 - x}$.

9. $\dfrac{5x^2 - 6x + 12}{x^3 - x^2 - 8x + 12}$.

10. $\dfrac{6}{x^3 + 1}$.

11. $\dfrac{7x^2 + 7x}{(3x^2 + 2)(2 - x)}$.

12. $\dfrac{3}{x^3 + 3x}$.

13. $\dfrac{3x^3 - 18x^2 + 4x - 4}{(2x^2 + 3)(x - 2)^2}$.

14. $\dfrac{2x^2 + 5x - 12}{(2x^2 + 3)(2x + 3)}$.

15. $\dfrac{x^3+1}{(x^2-2x+3)^2}$.

16. $\dfrac{x^2-x-12}{(2x^2-x-1)(x-2)^2}$.

17. $\dfrac{8x+6}{6x^2-x-1}$.

18. $\dfrac{8x^2+2}{(3x-2)(2x-3)^2}$.

19. $\dfrac{x^4-10x^2-1}{(x^2+1)^2(x+2)}$.

20. $\dfrac{3x^4+x-4}{(x^2+2)^2(x-3)}$.

21. $\dfrac{3x+15}{6x^2-5x-6}$.

22. $\dfrac{3x^3-2}{x^2-3x+2}$.

23. $\dfrac{6x^4-5x^3+6}{6x^2-5x-6}$.

24. $\dfrac{1}{(x^2+1)(x-2)^2}$.

25. $\dfrac{2x-1}{(x-2)^2}$.

26. $\dfrac{10x^2+2x-8}{(3x^2+2)(x+2)^2}$.

27. $\dfrac{2x^3-x^2}{(x^2+2)^2}$.

28. $\dfrac{2x-16}{(x-2)(x-3)(x+2)}$.

29. $\dfrac{2x^2-13x+15}{(x-2)^2(1-x)}$.

30. $\dfrac{8x^2+11x+24}{(2x-3)(x+3)(3x+2)}$.

31. $\dfrac{4x^2+3x-8}{(x^2+x+1)(2x+3)}$.

32. $\dfrac{x^4-x^2-3x+1}{2x^5-x^4+x^3}$.

33. $\dfrac{x^3-7x-2}{(x-1)(x^2+1)^2}$.

34. $\dfrac{3x^3+6x^2-23}{(x-2)^2(x+3)^2}$.

35. $\dfrac{7+10x+2x^2-x^3}{(x-2)^2(2x^2+1)}$.

36. $\dfrac{1+46x+x^2+14x^3-5x^4}{(x^2+2)^2(x-3)^2}$.

37. $\dfrac{8x^2-14x+3}{(2x^2-x+1)(2x-1)^2}$.

38. $\dfrac{x^2+2x-1}{x^4+5x^2+6}$.

39. $\dfrac{1}{x^2-a^2}$.

40. $\dfrac{2x^3-3x^2+10}{2x^2-x-6}$.

41. $\dfrac{x^4-x^2+2}{x^3(2x^2-x+1)}$.

42. $\dfrac{13+18x-4x^2}{(2x^2+3)(x-2)^2}$.

43. $\dfrac{4x^3+9x^2-x+9}{(x^2+1)^2(x-2)}$.

44. $\dfrac{x^2-31x+28}{(2x^2+3)(3x-2)^2}$.

CHAPTER XIX

Compound Interest and Annuities

98. SIMPLE INTEREST

There are numerous important applications of geometric progressions in business (interest), insurance (annuities), biology (laws of growth or multiplication), installment payments, and physics. We shall be interested mainly in those applications pertaining to finance problems. All such applications involve the idea of interest. ***Interest*** *is the amount of money paid at the* **end** *of a given interval of time for the use of a sum of money during that interval of time.* To illustrate, a man may desire the use of \$500 for the next 4 years and be willing to pay back \$600 at the end of the 4 years. *The* \$100 *which is the difference between the amount paid back and the amount borrowed is called interest.* The type and the rate of interest which the \$100 represents depends upon the choice of units.

Rate expressed as a decimal is based on the unit 1. *Rate expressed as per cent is rate per hundred.*

To determine what rate or part x is of y, determine the number of units in x (usually a fractional part of a unit) for every unit in y. Thus x is $\frac{x}{y}$ parts of y.

To determine what per cent x is of y, determine the number of units in x for every 100 units in y. Thus x is $x \div \frac{y}{100} = \frac{x}{y} \times 100$ per cent of y. Therefore

$$\frac{x}{y} = rate,$$

$$\frac{x}{y} \times 100 = per\ cent.$$

Thus we can either say 6 is $\frac{1}{4}$ of 24, or we can say 6 is 25% of 24.

The **rate of interest** *expressed as a decimal is the ratio of the interest earned in unit time to the principal. The rate of interest expressed as per cent is the rate per hundred.* Thus, if \$150 earns \$9 in one year the rate of interest is

$$\text{rate} = \tfrac{9}{150} = 0.06, \text{ or } 6\%.$$

For convenience, the unit of time is usually the year, although sometimes a fractional part of a year is used.

Simple interest *is the product of the principal, the rate of interest, and the number of units of time.*

The **amount** *at the end of a given period of time is the original principal plus the interest earned during this period of time.*

Let P equal the original principal, i the rate of interest per year expressed as a decimal, n the number of years, I the total interest, and A the amount.

We then have the following formulas for simple interest.

$$I = Pin, \tag{1}$$

$$A = P + I = P(1 + in). \tag{2}$$

Simple interest is usually for periods of 1 year or less, and is very rarely used for periods greater than 1 year. *By any quoted rate of interest* i *for a period which is a fractional part of a year is meant that the interest for this period is at the rate* i *per year.*

For periods of less than 1 year we shall, in this course, assume 30 days to a month and 1 month as $\frac{1}{12}$ of a year. When interest is calculated on the basis of 360 days in a year, the result is called **ordinary simple interest.** When interest is calculated by using the exact number of days and is based on 365 days in a year, the result is called **exact simple interest.**

Example 1. Find the interest and the amount of \$56 at simple interest for 8 months at 6%.

Solution. Here $P = \$56$, $n = \frac{8}{12}$, and $i = .06$.

By Equation (1)

$$I = Pin = \$56(\tfrac{2}{3})(0.06) = \$2.24.$$

By Equation (2)

$$A = P + I = \$56 + \$2.24 = \$58.24.$$

Example 2. What sum of money will amount to \$64 in 9 months at 4% simple interest?

Solution. Here $A = \$64$, $n = \frac{9}{12}$, and $i = .04$.

By Equation (2)

$$A = P(1 + in);$$

substituting, we have

$$\$64 = P(1 + .06 \times \tfrac{3}{4}) = P(1.045).$$

Solving for P gives

$$P = \frac{\$64}{1.045} = \$61.24.$$

99. COMPOUND INTEREST

It is customary for the lender of money and not the borrower to benefit at the end of a specified interval of time by the interest earned during this interval of time. Thus, the interest (simple interest) is either paid at the end of each specified equal interval of time, or the interest is added to the principal at the ends of equal intervals of time. The interval of time between two successive conversions of interest into principal is called the **conversion period.** This latter concept of computing interest is called compound interest. ***Compound interest*** *is the total accumulation of interest when calculated by adding to the principal at the end of each conversion period the interest (simple interest) earned during this conversion period, to form a new principal from which to calculate the interest (simple interest) for the next conversion period.*

If the conversion period is the year, we speak of the interest as being **compounded** or **convertible annually.** If the conversion period is 6 months, we speak of the interest as being **compounded** or **convertible semiannually,** and so forth.

Let P equal the original principal, i the rate of interest per year expressed as a decimal, n the number of years, m the number of conversion periods per year, and A the compound amount to which P accumulates at the end of n years.

If interest is compounded annually, the amount of the original principal P at the end of one year, or the principal for the beginning of the second year, is

$$P_1 = P(1 + i);$$

the amount at the end of two years is

$$P_2 = P_1(1 + i) = P(1 + i)^2;$$

the amount at the end of three years is

$$P_3 = P_2(1 + i) = P(1 + i)^3;$$

and so on until, finally, the amount A at the end of n years is

$$A = P(1 + i)^n. \tag{3}$$

Table V, pages 412–413, gives the values of $(1 + i)^n$ for different values of i and n.

The compound interest is the difference between the total amount and the initial principal; that is, the compound interest is $A - P$.

If interest is compounded m times a year, then Formula (3) becomes

$$A = P\left(1 + \frac{i}{m}\right)^{mn}. \tag{4}$$

This is evident since the conversion period is $\frac{1}{m}$ parts of the year, the interest rate per conversion period is $\frac{i}{m}$, and in n years there are mn conversion periods.

The principal P in Equation (3) can be called the present value of A. *The* ***present value*** *of an amount* A *which is payable* n *years from now is that principal* P *which will at compound interest amount to* A *dollars in* n *years.*

If interest is compounded annually at the rate i per cent per year, then the present value is found by solving Equation (3) for P. Hence,

$$P = \frac{A}{(1 + i)^n} = A(1 + i)^{-n}. \tag{5}$$

Table VII, pages 414–415, gives the values of $(1 + i)^{-n}$ for different values of i and n.

If interest is compounded m times per year at the rate i per cent per year, then the present value is found by solving Equation (4) for P. Hence,

$$P = A\left(1 + \frac{i}{m}\right)^{-mn} \tag{6}$$

It is very important in problems of this nature to ascertain whether the sum of money stated in a given problem refers to the present time or to some future time; that is, to observe whether the problem asks for the amount or whether it asks for the present value.

Example 1. At the time of the birth of a child the parents deposit $500 to his credit. What sum of money will the child have to his credit

on his eighteenth birthday if the money earns interest at the rate of 6%, compounded semiannually?

Solution. The \$500 refers to the present time, and the sum of money to be solved for refers to some future time. The amount A can be found by substituting $P = \$500$, $i = .06$, $m = 2$, and $n = 18$ in Equation (4). Thus,

$$A = P\left(1 + \frac{i}{m}\right)^{mn} = \$500(1.03)^{36}$$
$$= \$500(2.898278)$$
$$= \$1{,}449.14.$$

The value of $(1.03)^{36}$ is found on page 412, column 5, line 36. These problems can also be solved by logarithms.

Example 2. What sum of money will amount to \$1,000 in 15 years at 5%, compounded annually?

Solution. The sum of money to be found refers to the present time. The present value P can be found by substituting $A = \$1{,}000$, $i = .05$, and $n = 15$ in Equation (5). Thus,

$$P = A(1 + i)^{-n} = \$1{,}000(1.05)^{-15}$$
$$= \$1{,}000(0.481017)$$
$$= \$481.02.$$

The value of $(1.05)^{-15}$ is found on page 415, column 4, line 15.

100. EFFECTIVE RATE OF INTEREST

The **nominal rate** of interest is the quoted rate, or the so-called **yearly rate.** **Effective rate** of interest is the rate of interest which when compounded annually gives the interest that has been earned in one year. When interest is compounded annually, the effective rate of interest is the same as the nominal rate of interest. By definition the amount of P dollars at the end of one year at the effective rate e is the same as the amount of P dollars at the end of one year at the nominal rate i compounded m times a year. Therefore, by Equations (3) and (4), the relation between effective rate e and nominal rate i for $n = 1$ is given by the following equation:

$$P(1 + e) = P\left(1 + \frac{i}{m}\right)^{m},$$

or

$$1 + e = \left(1 + \frac{i}{m}\right)^{m}. \tag{7}$$

Solving for e gives

$$e = \left(1 + \frac{i}{m}\right)^m - 1. \tag{8}$$

Solving for i gives

$$i = m[(1 + e)^{1/m} - 1]. \tag{9}$$

Equation (8) is the formula for finding the effective rate e when the nominal rate i is given.

Equation (9) is the formula for finding the nominal rate i when the effective rate e is given.

Example 1. Find the effective rate equivalent to 4% compounded quarterly.

Solution. In this case $i = .04$ and $m = 4$. By Equation (8),

$$e = \left(1 + \frac{i}{m}\right)^m - 1;$$

substituting, we have

$$e = \left(1 + \frac{.04}{4}\right)^4 - 1;$$

simplifying gives

$$e = 0.406, \text{ or } 4.06\%.$$

Example 2. Find the nominal rate compounded monthly equivalent to the effective rate 6%.

Solution. In this case $e = .06$ and $m = 12$. By Equation (9),

$$i = m[(1 + e)^{1/m} - 1];$$

substituting, we have

$$i = 12[(1.06)^{1/12} - 1];$$

simplifying gives

$$i = 0.0584, \text{ or } 5.84\%.$$

Note: If the value of any part of the formulas cannot be found in the table, the problem can be solved by using logarithms.

EXERCISES

1. (a) 15 is what part of 20?
(b) 2.7 is what part of 54?
(c) 0.2 is what part of 0.5?
(d) $\frac{3}{4}$ is what part of $\frac{2}{3}$?

2. (a) 275 is what part of 350?
(b) 0.03 is what part of 1?
(c) 0.25 is what part of 7.5?
(d) $\frac{4}{5}$ is what part of $\frac{7}{8}$?

3. (a) 6 is what per cent of 12?
(b) 5 is what per cent of 4?
(c) 700 is what per cent of 350?

4. (a) 6 is what per cent of 5?
(b) 625 is what per cent of 800?
(c) 785 is what per cent of 1,200?

5. Find the simple interest on \$375 at 6% for 9 months.

6. Find the simple interest on \$1,250 at 5% for 4 months.

7. Find the simple interest on \$75 at 7% for 3 months.

8. Find the simple interest on \$100 at 6% for 60 days.

9. A bill of \$250 was not paid until 4 months after it was due. When paid, the bill amounted to \$254.17. What rate of interest was charged?

10. A man borrows \$200 from a bank, payable in 4 months at 6%. However, the banker deducts the \$4 interest in advance and gives the man \$196. At the end of 4 months the man pays the bank \$200. What rate of simple interest does he actually pay?

11. A bill for goods amounting to \$1,250 due in 90 days is presented to a bank for cash. If the bank charges 6% simple interest, how much cash will the bank allow the merchant?

12. A bank allowed a merchant \$991.73 on a bill of goods for \$1,000 due in 60 days. What rate of simple interest did the bank charge?

13. (a) Find the amount of \$3,000 for 12 years at 6%, compounded annually. (b) Find the amount of \$6,500 for 20 years at 5%, compounded annually.

14. (a) Find the amount of \$500 for 8 years at 6%, compounded semiannually. (b) Find the amount of \$1 for 10 years at 8%, compounded annually; semiannually; quarterly.

15. (a) Find the present value of \$1,000 due in 20 years, if money is worth 7%, compounded annually. (b) Find the present value of \$2,500 due in 25 years, if money is worth 6%, compounded semiannually.

16. (a) Find the present value of \$10,000 due in 12 years, if money is worth 8%, compounded quarterly. (b) Find the present value of \$800 due in 10 years, if money is worth 5%, compounded semiannually.

17. Find the effective rate of 6%, compounded (a) semiannually; (b) quarterly; (c) monthly.

18. Find the effective rate of 4%, compounded (a) semiannually; (b) quarterly; (c) monthly.

19. Find the effective rate of 5%, compounded (a) semiannually; (b) quarterly; (c) monthly.

20. Find the effective rate of 2%, compounded (a) semiannually; (b) quarterly, (c) monthly.

21. Find the nominal rate, compounded (a) semiannually, (b) quarterly, and (c) monthly, that is equivalent to an effective rate of 6%, of 4%, and of 2% in each case.

22. Find the nominal rate, compounded (a) semiannually, (b) quarterly, and (c) monthly, that is equivalent to an effective rate of 5%, of $4\frac{1}{2}\%$, and of 3% in each case.

23. From the American Experience Table of Mortality we find that out of 100,000 children who reach the age of 10 during any year, there are 99,251 living at the end of the year. What is the death rate for children aged 10?

24. Out of 92,637 individuals living at the age of 20 there are 89,032 living at the age of 25. What is the average death rate of young people between the ages of 20 and 25?

25. A utility company's gross income for a certain year is \$9,264,789 for electricity, \$4,275,874 for gas, and \$874,273 for steam heat. The income of each department is what per cent of the gross income?

26. The operating expenses of three departments A, B, and C are as follows: A, \$234.25, B, \$572.10, and C, \$50.75. Find what per cent each department's expense is of the total expense.

27. If the price of milk is raised one cent a quart and a certain milk company's business falls from 65,249 quarts daily to 64,375 quarts, what is the per cent loss in trade?

28. If one company steps up production from 40,000 units to 45,000 units per day, and a second company steps up production from 15,000 units to 20,000 units per day, compare their per cent of increase in production. How many units per day must the second company produce to have the same rate of increase in production as the first company?

29. If the following table represents the gross income of different companies for two successive years, find the per cent of increase or decrease in gross income in each case.

1929	1930
\$2,125,600	\$1,620,200
978,320	990,250
572,653	898,275
975,000	875,000
375,000	275,000
75,250	752,500

30. What sum of money will amount to \$10,000 in 20 years if money is worth 7%, compounded semiannually?

31. A man buys a lot for \$1,200. What should be the selling price of the lot 10 years hence if money is worth 6%, convertible annually?

32. Which is the better proposition, \$500 cash settlement, or \$1,000 settlement 10 years hence, if money is (a) worth 6%, compounded annually? (b) worth 8%, compounded annually?

33. Find the time required for money to double itself at 6%; at 5%; at 6%, compounded semiannually; at 5%, compounded semiannually.

34. The following are equivalent to what rates of interest? (a) \$100 cash or \$120 in 60 days. (b) \$80 cash or \$100 in 90 days. (c) \$450 cash or \$500 in 30 days.

35. A man holds a noninterest-bearing note for \$1,000 payable in 8 years. What is the cash value of the note if money is worth 6%, convertible semiannually?

101. ANNUITIES

We are all familiar with experiences of making equal payments at the end of equal intervals of time, such as rent, insurance, payments on a home or debt, installment payments, as well as savings. All such series of payments are called annuities. An ***annuity*** *is a series of equal payments made at equal intervals of time.* In this course we are primarily interested in those cases where the payments are made at the end of each interval of time.

The **amount of an annuity** at the end of a given period of time is the sum to which the various payments accumulate at compound interest.

The symbol $s_{\overline{n}|}$, read "s angle n," is used to represent the amount of an annuity of \$1 paid at the end of each year for n years, when interest is compounded annually. To find the value of $s_{\overline{n}|}$, observe that the first dollar will be on interest for $n - 1$ years and by Equation (3), Article 99, will amount to $(1 + i)^{n-1}$; the second dollar will be on interest for $n - 2$ years and will amount to $(1 + i)^{n-2}$; the third dollar will be on interest for $n - 3$ years and will amount to $(1 + i)^{n-3}$; and so on; finally the $(n - 1)$st dollar will be on interest for 1 year and will amount to $(1 + i)$, and the nth or last dollar is a cash payment made at the end of the nth year and its amount is 1. The amount of this annuity is the sum of the amounts of the n payments. Thus, *the amount* $s_{\overline{n}|}$ *of an annuity* of \$1 *per year for* n *years at a rate of interest* i *compounded annually is found by taking the sum*

$$s_{\overline{n}|} = (1 + i)^{n-1} + (1 + i)^{n-2} + (1 + i)^{n-3} + \cdots + (1 + i)^2 + (1 + i) + 1. \tag{10}$$

Rewriting,

$$s_{\overline{n}|} = 1 + (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-3} + (1 + i)^{n-2} + (1 + i)^{n-1}.$$

These terms form a geometric progression in which $a = 1$, $r = (1 + i)$, and $n = n$. From Equation (4), Article 84, Chapter XIV, this sum is

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}. \tag{11}$$

Table VII, pages 416–417, gives the values of $s_{\overline{n}|}$ for different values of n and i.

The amount of an annuity of R dollars per year is R times the amount of an annuity of \$1 per year; that is,

$$A = Rs_{\overline{n}|}. \tag{12}$$

The present value of an annuity is the sum of the present values of all the future payments.

The symbol $a_{\overline{n}|}$, read "a angle n," is used to represent the present value of an annuity of \$1 paid at the end of each year for n years when interest is compounded annually. The first dollar is payable 1 year hence and its present value by Equation (5), Article 99, is $(1 + i)^{-1}$; the second dollar is payable 2 years hence and its present value is $(1 + i)^{-2}$; and so forth; finally the $(n - 1)$st dollar is payable $(n - 1)$ years hence and its present value is $(1 + i)^{-(n-1)}$; and the nth or last dollar is payable n years hence and its present value is $(1 + i)^{-n}$. The present value of this annuity is the sum of the present values of the n payments. Thus, *the present value* $a_{\overline{n}|}$ *of an annuity of* \$1 *per year for* n *years at a rate of interest* i *compounded annually is found by taking the sum*

$$a_{\overline{n}|} = (1+i)^{-1} + (1+i)^{-2} + \cdots + (1+i)^{-(n-1)} + (1+i)^{-n}. \tag{13}$$

Rewriting,

$$a_{\overline{n}|} = (1+i)^{-n} + (1+i)^{-(n-1)} + \cdots + (1+i)^{-2} + (1+i)^{-1}.$$

These terms form a geometric progression in which $a = (1 + i)^{-n}$, $r = (1 + i)$, and $n = n$. From Equation (4), Article 84, this sum is

$$a_{\overline{n}|} = \frac{(1+i)^{-n}[(1+i)^n - 1]}{(1+i) - 1} = \frac{1 - (1+i)^{-n}}{i}. \tag{14}$$

Table VIII, pages 418–419, gives the values of $a_{\overline{n}|}$ for different values of n and i.

The present value of an annuity of R dollars per year is R times the present value of an annuity of \$1 per year; that is,

$$P = Ra_{\overline{n}|}. \tag{15}$$

Here, as in Article 99, it is necessary to ascertain whether the sum of money stated in a given problem refers to the present time or to some future time; that is, to determine whether the given problem is a present value problem or an amount problem.

Example 1. An annual savings of \$100 will amount to how much in 20 years if money is worth 6%, compounded annually?

Solution. This is an amount problem in which $R = \$100$, $n = 20$, and $i = .06$. Hence, by Equation (12),

$$\begin{aligned} A = Rs_{\overline{20}|} &= \$100\,\frac{(1.06)^{20} - 1}{0.06} \\ &= \$100(36.785591) \\ &= \$3{,}678.56. \end{aligned}$$

The value of $s_{\overline{20}|}$ is found in page 417, column 5, line 20.

Example 2. A man buys a house for \$9,500. He pays \$1,500 in cash and wishes to pay the remainder in 10 equal annual installments. Find the annual payments if money is worth 6%, compounded annually.

Solution. Clearly the \$8,000 remaining due must be the present value of the 10 equal payments. In this case $P = \$8{,}000$, $n = 10$, and $i = .06$. The annual payments R are found by solving the following equation for R:

$$\$8{,}000 = Ra_{\overline{10}|}.$$

Hence,

$$\begin{aligned} R &= \$8{,}000(a_{\overline{10}|})^{-1} \\ &= \$8{,}000(0.135868) \\ &= \$1{,}086.94. \end{aligned}$$

If we do not have a table of values for $a_{\overline{n}|}^{-1}$ or $a_{\overline{n}|}$, then it is necessary to solve the equation

$$\$8{,}000 = R\,\frac{1 - (1.06)^{-10}}{0.06},$$

for R. This can be done with the aid of a table of logarithms.

If we have a table of values of $(1 + i)^{-n}$, we find that $(1.06)^{-10} = 0.5583948$. Hence,

$$\$8{,}000 = R\,\frac{1 - (1.06)^{-10}}{0.06} = R\,\frac{1 - 0.5583948}{0.06}$$

$$= R\,\frac{0.4416052}{0.06} = (7.36008)R.$$

Solving for R,

$$R = \$1{,}086.94.$$

Example 3. Find the present value of a \$1,000 bond, maturing in 12 years, bearing interest at 4%, payable annually, if money is worth 6%.

Solution. Here, the annual interest of \$40 from the bond constitutes an annuity whose present value at 6% is

$$P_1 = \$40 a_{\overline{12}|} = \$40(8.383844) = \$334.55.$$

The present value of the principal, \$1,000, which is payable 12 years hence, is

$$P_2 = \$1{,}000(1.06)^{-12} = \$1{,}000(0.496969) = \$496.97.$$

The present value of the bond is the sum of the two present values:

$$P_1 + P_2 = \$831.52.$$

If money is compounded m times a year and the payments are made m times a year, then the problem is equivalent to mn conversion periods at rate $\frac{i}{m}$ per period. Thus, i should be replaced by $\frac{i}{m}$ and $(1 + i)$ by $\left(1 + \frac{i}{m}\right)^m$ in Formulas (11) and (14). These formulas then become

$$s_{\overline{mn}|} = \frac{\left(1 + \frac{i}{m}\right)^{mn} - 1}{\frac{i}{m}}, \tag{16}$$

and

$$a_{\overline{mn}|} = \frac{1 - \left(1 + \frac{i}{m}\right)^{-mn}}{\frac{i}{m}}. \tag{17}$$

A treatment of the more general case, that is, the case in which the periods of conversion are different from the periods of payments, may be found in any textbook on finance. However, the formulas are

The amount $\boldsymbol{A}$ *at the end of* $\boldsymbol{n}$ *years of* $\boldsymbol{R}$ *dollars per year, payable in* $\boldsymbol{p}$ *equal installments at the end of equal intervals during each year, with interest compounded* $\boldsymbol{m}$ *times a year at the rate* $\boldsymbol{i}$ *per year, is*

$$\boldsymbol{A = Rs_{\overline{n}|}^{(p)} = \frac{R}{p} \frac{\left(1 + \frac{i}{m}\right)^{mn} - 1}{\left(1 + \frac{i}{m}\right)^{\frac{m}{p}} - 1}}. \tag{18}$$

The present value $\boldsymbol{P}$ *of* $\boldsymbol{R}$ *dollars per year, payable in* $\boldsymbol{p}$ *equal installments at the end of equal intervals during each year for* $\boldsymbol{n}$

years, with interest compounded m *times a year at the rate* i *per year, is*

$$P = Ra_{\overline{n}|}^{(p)} = \frac{R}{p} \frac{1 - \left(1 + \frac{i}{m}\right)^{-mn}}{\left(1 + \frac{i}{m}\right)^{\frac{m}{p}} - 1}. \tag{19}$$

EXERCISES

Note: If the length of the conversion period is not given it is always understood to be a year.

1. An annual savings of \$500 will amount to how much in 20 years if money is worth 6%?

2. Find the amount of an annuity of \$1,000 paid at the end of each year for 25 years at 5%, compounded annually.

3. What annual savings are necessary to amount to \$10,000 in 10 years if money is worth 5%?

4. Find the annual payment necessary to amount to \$8,500 in 20 years if money is worth 4%, compounded annually.

5. A father wishes to form a trust fund so that his son will receive \$800 at the end of each year for the next eight years. What sum must he place in the trust fund if money is worth 4%, compounded annually?

6. What is the present value of an annuity of \$500 paid at the end of each year for 25 years if money is worth 6%, compounded annually?

7. What sum of money must be set aside annually at 6% to provide for the payment of a noninterest-bearing debt of \$1,500 due in 8 years?

8. What sum of money must be set aside annually at 6% to provide for the payment of a note of \$1,500 due in 8 years if the note bears interest at 6%?

9. What equal deposits at the end of each year for 10 years will amount to \$1,200 if money is worth 4%, compounded annually?

10. A man 40 years of age deposits \$10,000 in a savings account which pays 5% interest, compounded annually. What equal amounts can he withdraw annually for 10 years, beginning on his sixty-fifth birthday?

11. A city has bonds to the amount of \$200,000 falling due in 15 years. What sum of money must be set aside semiannually in a sinking fund to purchase the bonds when they fall due if the money in the sinking fund accumulates at 5%, convertible semiannually?

12. It is estimated that in 10 years a piece of machinery must be replaced by new machinery which will cost \$6,000. What annual savings are necessary to meet this need if the savings earn 6%, compounded annually?

13. A man wishes his family to receive \$1,266.50 per year for 20 years after his death. What amount of insurance must the man carry if the

insurance company pays interest at the rate of $3\frac{1}{2}\%$, compounded annually?

14. A building valued at \$2,500,000 has an expected life of 40 years. What is the yearly depreciation if money is worth 6%?

15. A man agrees to buy a house for \$1,000 in cash and equal payments of \$1,000 per year for 10 years. What is the purchasing price of the house if money is worth 7%?

16. Annual payments of \$600 each for 12 years will pay off a mortgage of what amount if money is worth 6%, compounded annually?

17. The special assessment on a piece of property is \$35.50 per year for 10 years. What should be the cash settlement price if money is worth 5%?

18. What is the purchase price of a \$1,000 bond maturing in 15 years, bearing interest at 6%, payable semiannually, if money is worth 5%, convertible semiannually?

19. If an automobile that costs \$1,495 when new is worth \$200 at the end of 6 years, what is the depreciation for each 6 months if money is worth 6%, convertible semiannually?

20. A man agrees to pay \$200 every 6 months for 5 years to meet his obligation. If money is worth 6%, compounded semiannually, what is the amount of the obligation?

21. The premiums on a \$1,000 twenty-year endowment insurance policy are \$40 per year paid at the end of each year. What amount of the premiums must the insurance company set aside annually to mature the policy if money earns 4% annually?

22. What annual rent should be charged for a house to net the owner 6% on his investment if the house cost \$8,500 and the repairs average \$100 per year? Assume that the house will be worth \$5,000 at the end of 15 years and that money is worth 6%, compounded annually.

23. An equal monthly savings of \$10 each will amount to how much in 12 years if money is worth 4%, compounded semiannually?

24. What is the present value of an annuity of \$20 per month for 25 years if money is worth 5%, compounded quarterly?

25. A father wishes to build a fund to assure a college education for his son. What sum of money must he deposit on each of the child's birthdays for 18 years so that the son, beginning on his nineteenth birthday, can withdraw \$1,000 per year for four years, if the money in the fund earns 6% interest, compounded annually?

26. What equal monthly payments are necessary to pay off a mortgage of \$8,000 in 12 years if money is worth 6%, compounded semiannually?

27. A man makes a cash payment of \$1,000 on a house and agrees to pay \$50 each month for 10 years. If money is worth 6%, compounded semiannually, what is the purchasing price of the house?

28. Prove that $c_{\overline{n}|} = s_{\overline{n}|}(1 + i)^{-n}$.

CHAPTER XX

Mathematical Induction

102. MATHEMATICAL INDUCTION

Many mathematical formulas and scientific laws involving a variable positive integer can be proved by a method called mathematical induction. The method consists of the following four parts.

PART (1). *Either discover a formula to prove true, or prove a given formula true.*

PART (2). *Verify the formula to see whether it is true for* $n = 1$, *and perhaps for the next few integers.* This is accomplished by substituting for the variable number n, in both members of the given equation, the values 1, 2, 3, and so on, one by one.

PART (3). *Assume that the formula is true for the first* $\boldsymbol{k}$ *integers, and in particular is true for* $\boldsymbol{n = k}$.

The formula is either true for all integers, or the formula will fail for some particular integer. Thus,

PART (4). *Prove that the formula is either true or false for* $\boldsymbol{n = k + 1}$ *by the aid of the assumption that the formula is true for* $\boldsymbol{n = k}$.

We should not make the assumption in Part (3) without verifying that the formula is true for at least $n = 1$. To illustrate, if we assume the erroneous statement that the sum of the first k even integers is an odd number, then we can prove that the sum of any number of even integers is an odd number, a statement that is false. However, if we try to verify the assumption we find that the sum of the first one, two, or three even integers is an even number and not an odd number. Thus, Part (2) cannot be omitted in any complete proof by mathematical induction.

If in Part (4) we succeed in proving the formula is true for $n = k + 1$, then we have proved that there does not exist a value for k such that the formula fails for $n = k + 1$. Hence, the formula must be true for all integral values of n. If in Part (4)

we do not succeed in proving that both members of the given equation are identical for $n = k + 1$, then there exists a value for k such that the formula is true only for integers equal to or less than k.

Example 1. Find the sum of the first n odd integers.

Solution. In this case let us attempt to discover a formula for the sum of the first n odd integers. To do this, let us find the sum of the first two odd integers, the first three odd integers, and so forth. Thus,

$$\begin{aligned} 1 &= 1 = 1^2, \\ 1 + 3 &= 4 = 2^2, \\ 1 + 3 + 5 &= 9 = 3^2, \\ 1 + 3 + 5 + 7 &= 16 = 4^2. \end{aligned}$$

From this we discover that

the sum of the first two odd integers is 2^2,
the sum of the first three odd integers is 3^2,
the sum of the first four odd integers is 4^2.

We might infer from this that

the sum of the first n *odd integers is* n^2.

In this example, for Part (1) we have discovered a formula we wish to prove true; that is, we wish to prove the formula

$$1 + 3 + 5 + 7 + \cdots + (2n - 1) = n^2.$$

Observe that the nth even integer is $2n$, and hence the nth odd integer is $2n - 1$; that is, the nth term is $2n - 1$. Also observe that the nth term must be an expression such that when $n = 1$ the expression gives the first term, when $n = 2$ it gives the second term, when $n = 3$ it gives the third term, and so on.

For Part (2) the formula was verified for $n = 1$, $n = 2$, $n = 3$, and $n = 4$. This was done in the process of discovering the formula.

For Part (3) assume that the formula is true for the first k integral values of n and in particular is true for $n = k$; that is, after substituting k for n in the given formula, assume that

$$1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2.$$

For Part (4) it is necessary to prove whether or not the sum of the first $(k + 1)$ odd integers is $(k + 1)^2$, on the assumption that the sum of the first k odd integers is k^2; that is, it is necessary to prove whether or not the formula is true for $n = k + 1$. Thus, after substituting $(k + 1)$ for n in both members of the given formula, it is necessary to prove whether or not

$$1 + 3 + 5 + 7 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2$$

on the assumption that

$$1 + 3 + 5 + 7 + \cdots + (2k - 1) = k^2.$$

Thus, prove whether or not

$$k^2 + (2k + 1) = (k + 1)^2.$$

This is an identity since

$$k^2 + 2k + 1 = (k + 1)^2.$$

This completes the proof, since the formula was verified and found to hold for $n = 1$, 2, 3, and 4, respectively, and it was shown that if the formula holds for $n = k$, it does not fail for $n = k + 1$. That is, there does not exist an integral value of k for which the formula fails for $n = k + 1$; hence the formula must be true for all integral values of n.

Since the terms in Example 1 form an arithmetic progression, in which $a = 1$, $d = 2$, and $n = n$, it is possible in this case to find the sum of the series by using the formulas given by Equations (1) and (2), Article 83, Chapter XIV. The nth term is

$$l = a + (n - 1)d = 1 + (n - 1)2 = 2n - 1,$$

and the sum is

$$s = \frac{n}{2}(a + l) = \frac{n}{2}(1 + 2n - 1) = n^2.$$

Example 2. Prove that

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3 = \frac{n^2}{4}(n + 1)^2.$$

Solution. In this case, for Part (1) the formula is given. For Part (2), let us verify the formula.

$$\text{For } n = 1,\ 1^3 = 1 = \frac{1^2}{4}(1 + 1)^2,$$

$$\text{For } n = 2,\ 1^3 + 2^3 = 9 = \frac{2^2}{4}(2 + 1)^2,$$

$$\text{For } n = 3,\ 1^3 + 2^3 + 3^3 = 36 = \frac{3^2}{4}(3 + 1)^2,$$

$$\text{For } n = 4,\ 1^3 + 2^3 + 3^3 + 4^3 = 100 = \frac{4^2}{4}(4 + 1)^2.$$

Thus, the formula is satisfied for $n = 1$, 2, 3, and 4.

For Part (3), assume that the formula is true for the first k integers and in particular is true for $n = k$. That is, after substituting k for n in both members of the given formula, assume that

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 = \frac{k^2}{4}(k + 1)^2.$$

For Part (4) it is necessary to prove whether or not the formula is true for $n = k + 1$; that is, after substituting $k + 1$ for n in both members of the given formula, it is necessary to prove whether or not

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 + (k+1)^3 = \frac{(k+1)^2}{4}(k+1+1)^2$$

on the assumption that

$$1^3 + 2^3 + 3^3 + 4^3 + \cdots + k^3 = \frac{k^2}{4}(k+1)^2.$$

Thus, prove whether or not

$$\frac{k^2}{4}(k+1)^2 + (k+1)^3 = \frac{(k+1)^2}{4}(k+2)^2.$$

Factoring out $\frac{(k+1)^2}{4}$, we have,

$$\frac{(k+1)^2}{4}[k^2 + 4(k+1)] = \frac{(k+1)^2}{4}(k+2)^2,$$

which completes the proof, since this is an identity. Therefore the formula is true for all integral values of n, since the formula does not fail for $n = k + 1$ for any value of k, and the formula is true for $k = 1$, 2, 3, and 4.

Example 3. Let us attempt to prove the formula

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + n(n+1) = 3n^2 - 3n + 2.$$

Solution. Let us verify the formula for $n = 1$, 2, and 3.

For $n = 1$, $1 \cdot 2 = 2 = 3 - 3 + 2$,
For $n = 2$, $1 \cdot 2 + 2 \cdot 3 = 8 = 12 - 6 + 2$,
For $n = 3$, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 = 20 = 27 - 9 + 2$.

Thus the formula is true for $n = 1$, 2, and 3, respectively.

Let us assume that the formula holds for the first k integers and thus, after substituting k for n in both members of the given equation, assume that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + k(k+1) = 3k^2 - 3k + 2.$$

Then after substituting $k + 1$ for n in both members of the given equation, let us attempt to prove that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) + (k+1)(k+2) = 3(k+1)^2 - 3(k+1) + 2;$$

that is, to prove that

$$3k^2 - 3k + 2 + (k+1)(k+2) = 3(k+1)^2 - 3(k+1) + 2.$$

Simplifying both members gives

$$4k^2 + 4 = 3k^2 + 3k + 2.$$

Now this equation is not an identity and hence is not true for all values of k k. This is a conditional equation and is true for only those values of k that will satisfy this equation. Transposing and collecting, we have

$$k^2 - 3k + 2 = 0;$$

factoring, gives

$$(k - 1)(k - 2) = 0;$$

solving for k, we obtain

$$k = 1 \qquad \text{or} \qquad k = 2.$$

Thus, the given formula is true for $n = k + 1$ only if $k = 1$ or $k = 2$; that is, the given formula fails and is not true for any value of n greater than $2 + 1 = 3$. If we try to verify the formula for $n = 4$, we will find that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 = 40,$$

while

$$3n^2 - 3n + 2 = 48 - 12 + 2 = 38,$$

which shows that the formula fails for $n = 4$.

In the study of series it is very important to be able to discover the law of formation of a given series when a few of its terms are given. The discovery of such a law is necessary before the general term (the nth term) can be written. To illustrate, the nth term of the series

$$1 + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 3^2} + \frac{1}{4 \cdot 3^3} + \frac{1}{5 \cdot 3^4} + \cdots$$

is

$$\frac{1}{n \cdot 3^{n-1}}.$$

EXERCISES

Write the nth term, the kth term, and the $(k + 1)$st term in each of the following series (Exercises 1 through 14).

1. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots$.

2. $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$.

3. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$.

4. $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots$.

5. $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[5]{2}} + \cdots$.

6. $\frac{1 \cdot 2}{2 \cdot 3} + \frac{2 \cdot 3}{3 \cdot 4} + \frac{3 \cdot 4}{4 \cdot 5} + \frac{4 \cdot 5}{5 \cdot 6} + \cdots$.

7. $\frac{1 \cdot 2}{3 \cdot 5} + \frac{2 \cdot 3}{5 \cdot 7} + \frac{3 \cdot 4}{7 \cdot 9} + \frac{4 \cdot 5}{9 \cdot 11} + \cdots$.

8. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$.

9. $\frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{6} + \frac{x^4}{8} + \cdots$.

10. $x + \frac{2x^2}{2!} + \frac{3x^3}{3!} + \frac{4x^4}{4!} + \cdots$.

11. $1 - 4x + 9x^2 - 16x^3 + \cdots$.

12. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$.

13. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$.

14. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$.

Find the first four terms of each of the series whose nth terms are given in Exercises 15 through 23.

15. $\frac{n+1}{n!}$.

16. $\frac{n}{(2n-1)}$.

17. $\frac{n}{2^n}$.

18. $\frac{(n+1)^2}{n(n+2)}$.

19. $\frac{n(n+1)^2}{(2n-1)!}$.

20. $\frac{2n+1}{n^2+1}$.

21. $\frac{(-1)^n 3 \cdot n^2}{2^n(2n-1)}$.

22. $\frac{n \cdot 2^{n+1}}{(2n)!}$.

23. $\frac{n(n+1)}{3^{n-1}(2n-1)^2}$.

Prove each of the following (Exercises 24 through 38) by mathematical induction.

24. $1 + 2 + 3 + 4 + \cdots + n = \frac{n}{2}(n+1)$.

25. $2 + 4 + 6 + 8 + \cdots + 2n = n(n+1)$.

26. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n}{6}(n+1)(2n+1)$.

27. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$.

28. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

29. $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \frac{n}{6}(n+1)(2n+7)$.

30. $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$.

31. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n}{3}(n+1)(n+2)$.

32. $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$.

33. $2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = \frac{2}{3}n(n+1)(2n+1)$.

34. $2^3 + 4^3 + 6^3 + \cdots + (2n)^3 = 2n^2(n+1)^2$.

35. $1 + 5 + 9 + \cdots + (\quad) = n(2n-1)$.

36. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + (\quad) = 1 - \frac{1}{2^n}$.

37. $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + (\quad) = \frac{n}{2n+1}$.

38. $1^4 + 2^4 + 3^4 + \cdots + (\quad)$

$$= \frac{n}{30}(n+1)(2n+1)(3n^2 + 3n - 1).$$

39. Prove that $x^n - y^n$ contains $x - y$ as a factor if n is a positive integer.
Hint: Make use of the identity $x^{k+1} - y^{k+1} = x(x^k - y^k) + y^k(x - y)$.

103. PROOF OF THE BINOMIAL THEOREM FOR POSITIVE INTEGRAL EXPONENTS BY MATHEMATICAL INDUCTION

Let us prove the binomial theorem for positive integers by mathematical induction; that is, let us prove the formula

$$(a+x)^n = a^n + {}_nC_1 a^{n-1}x + {}_nC_2 a^{n-2}x^2 + \cdots + {}_nC_{r-1}a^{n-r+1}x^{r-1} + \cdots + x^n. \qquad (1)$$

By direct multiplication it can be verified that

$$\text{for } n = 1,\ (a+x)^1 = a + x,$$
$$\text{for } n = 2,\ (a+x)^2 = a^2 + {}_2C_1 ax + {}_2C_2 x^2 = a^2 + 2ax + x^2.$$

Let us assume that the formula is true for the first k integers, and in particular is true for $n = k$. Thus, assume that

$$(a+x)^k = a^k + {}_kC_1 a^{k-1}x + {}_kC_2 a^{k-2}x^2 + \cdots + {}_kC_{r-2}a^{k-r+2}x^{r-2} + {}_kC_{r-1}a^{k-r+1}x^{r-1} + \cdots + x^k \qquad (2)$$

and then prove that the formula is true for $n = k + 1$. That is, prove that

$$(a+x)^{k+1} = a^{k+1} + {}_{k+1}C_1 a^k x + {}_{k+1}C_2 a^{k-1}x^2 + \cdots + {}_{k+1}C_{r-1}a^{k-r+2}x^{r-1} + \cdots + x^{k+1}. \qquad (3)$$

To prove this, let us multiply each term of Equation (2) by $(a + x)$. We then have

$$(a + x)^{k+1} = a^{k+1} + {}_kC_1a^kx + {}_kC_2a^{k-1}x^2 + \cdots + {}_kC_{r-2}a^{k-r+3}x^{r-2} + {}_kC_{r-1}a^{k-r+2}x^{r-1} + \cdots + ax^k + a^kx + {}_kC_1a^{k-1}x^2 + \cdots + {}_kC_{r-2}a^{k-r+2}x^{r-1} + \cdots + x^{k+1};$$

adding like terms we obtain

$$(a + x)^{k+1} = a^{k+1} + ({}_kC_1 + 1)a^kx + ({}_kC_2 + {}_kC_1)a^{k-1}x^2 + \cdots + ({}_kC_{r-1} + {}_kC_{r-2})a^{k-r+2}x^{r-1} + \cdots + x^{k+1}. \quad (4)$$

Let us simplify each coefficient of Equation (4) separately.

$${}_kC_1 + 1 = k + 1 = {}_{k+1}C_1.$$

$${}_kC_2 + {}_kC_1 = \frac{k(k-1)}{1\cdot 2} + \frac{k}{1} = k\left(\frac{k-1}{2} + 1\right)$$

$$= \frac{k(k+1)}{1\cdot 2} = {}_{k+1}C_2.$$

$${}_kC_{r-1} + {}_kC_{r-2} = \frac{k(k-1)\cdots(k-r+3)(k-r+2)}{1\cdot 2\cdots(r-2)(r-1)} + \frac{k(k-1)\cdots(k-r+3)}{1\cdot 2\cdots(r-2)}$$

$$= \frac{k(k-1)\cdots(k-r+3)}{1\cdot 2\cdots(r-2)}\left[\frac{k-r+2}{r-1} + 1\right]$$

$$= \frac{k(k-1)\cdots(k-r+3)}{1\cdot 2\cdots(r-2)}\left[\frac{(k+1)}{(r-1)}\right]$$

$$= \frac{(k+1)k(k-1)\cdots(k-r+3)}{1\cdot 2\cdot 3\cdots(r-1)}$$

$$= {}_{k+1}C_{r-1}.$$

Therefore, Equation (4) becomes

$$(a + x)^{k+1} = a^{k+1} + {}_{k+1}C_1a^kx + {}_{k+1}C_2a^{k-1}x^2 + \cdots + {}_{k+1}C_{r-1}a^{k-r+2}x^{r-1} + \cdots + x^{k+1},$$

which is the same as Equation (3). This completes the proof, since we have proved that the formula does not fail for $n = k + 1$ since we have proved that the formula does not fail for any value of k, and we have verified that the formula is true for $k = 1$ and 2.

CHAPTER XXI

Infinite Series

104. INFINITE SERIES

The sum of a finite series in which all the terms are finite is finite.

In the case of infinite series (unending series) there may exist a number that is greater than the numerical value of the sum of any number of terms, no matter how many are taken, or it may happen that it is possible to find enough terms whose sum is numerically greater than any number you please. Thus the following classification of infinite series may be given. Let

$$u_1 + u_2 + u_3 + u_4 + \cdots$$

be an infinite series, and let

$$S_1 = u_1,$$
$$S_2 = u_1 + u_2$$
$$S_3 = u_1 + u_2 + u_3,$$

and so on; finally

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n. \tag{1}$$

That is, let S_n be the sum of the first n terms. As the number of terms is allowed to increase indefinitely, one of the following conditions exists.

S_n approaches some number as a limit,

or

S_n increases numerically without bound,

or

S_n is bounded but does not approach a limit.

The series in the first case are called convergent series, and the series in the latter two cases are called divergent series. Thus, we have the

DEFINITION. *If in an infinite series the sum of the first* **n** *terms approaches a limit as* **n** *increases indefinitely, the series is said to be* ***convergent*** *and the limit of the sum of the first* **n** *terms is called the* ***sum*** *of the series. All other infinite series are called* ***divergent series*** *and their sum does not exist.* That is,

$$\text{if } \lim_{n\to\infty} S_n = S, \text{ the series is convergent,} \tag{2}$$

$$\text{if } \lim_{n\to\infty} S_n \text{ does not exist the series is divergent.} \tag{3}$$

We read $\lim_{n\to\infty} S_n$, "the limit of S_n as n grows larger without bound."

ILLUSTRATION 1. The series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} \cdots$$

is a convergent series because S_n approaches some number as a limit as n grows larger without bound. The terms of this series form a geometric progression with $a = 1$ and $r = \frac{1}{2}$. For $n = n$, the sum S_n, by Equation (4), Article 84, Chapter XIV, is

$$S_n = \frac{1\left[1 - \left(\frac{1}{2}\right)^n\right]}{1 - \frac{1}{2}} = 2 - \left(\frac{1}{2}\right)^{n-1}.$$

Therefore,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[2 - \left(\frac{1}{2}\right)^{n-1}\right] = 2.$$

ILLUSTRATION 2. The series

$$1 + 2 + 3 + 4 + 5 + \cdots$$

is a divergent series because S_n increases without bound as n grows larger without bound. The terms of this series form an arithmetic progression with $a = 1$, and $d = 1$. For $n = n$, the sum S_n, by Equation (2), Article 83, is

$$S_n = \frac{n}{2}(1 + n).$$

Therefore,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n}{2}(1 + n) = \infty.$$

ILLUSTRATION 3. The series

$$1 - 1 + 1 - 1 + 1 - \cdots$$

is a divergent series because S_n does not approach a limit as n increases indefinitely, but oscillates between 1 and 0. Such series are sometimes called **oscillating series.**

Example. Determine whether the infinite series

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots$$

is convergent or divergent.

Solution. The terms of this infinite series form a geometric progression with ratio r. By Equation (4), Article 84, the sum of the first n terms is

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

Since $\lim_{n\to\infty} r^n = 0$ if $|r| < 1$, and $\lim_{n\to\infty} r^n = \infty$ if $|r| > 1$, it follows that

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r}$$

if $|r| < 1$, and

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(r^n - 1)}{r - 1} = \infty$$

if $|r| > 1$. Therefore from definition the given series converges if $|r| < 1$ and diverges if $|r| > 1$.

There are several methods used for determining whether a series is convergent or divergent. We shall give only two methods, the **comparison test** and the **ratio test.**

105. COMPARISON TEST FOR CONVERGENCE AND DIVERGENCE

We shall prove two theorems that can be used to prove the convergence or divergence of certain series by comparing the given series with a series that is known to be convergent or divergent. We shall also give without proof a few theorems pertaining to convergent and divergent series.

Comparison Test for Convergence

THEOREM I. *If each term of a given series of positive terms to be tested, such as*

$$u_1 + u_2 + u_3 + u_4 + \cdots,$$

from the ***k****th term (that is, any term) on is equal to or less than the corresponding term of a known convergent series of positive terms,*

such as

$$v_1 + v_2 + v_3 + v_4 + \cdots,$$

then the given **u***-series is convergent.*

Proof: If

$$u_k \leqq v_k,$$
$$u_{k+1} \leqq v_{k+1},$$
$$u_{k+2} \leqq v_{k+2},$$

and so on, and finally

$$u_n \leqq v_n,$$

then it follows that the sum

$$U_n = u_k + u_{k+1} + u_{k+2} + \cdots + u_n \tag{4}$$

is less than or at most equal to the sum

$$V_n = v_k + v_{k+1} + v_{k+2} + \cdots + v_n. \tag{5}$$

Since the v series is known to converge, then by definition the limit of V_n, as n increases indefinitely, exists. Since $U_n \leqq V_n$, the limit of U_n must exist and is equal to or less than the limit of V_n.

The sum of the first $k - 1$ terms of the series to be tested is

$$S_{k-1} = u_1 + u_2 + u_3 + u_4 + \cdots + u_{k-1}.$$

Since this is the sum of a finite number of terms, the sum S_{k-1} must be finite.

Since

$$S_n = S_{k-1} + U_n,$$

then,

$$\lim_{n\to\infty} S_n = S_{k-1} + \lim_{n\to\infty} U_n.$$

It follows from this that the limit of S_n exists and is finite since it was shown that the limit of U_n exists. Therefore, the given infinite series

$$u_1 + u_2 + u_3 + u_4 + \cdots,$$

is a convergent series.

Comparison Test for Divergence

THEOREM II. *If each term of a given series of positive terms to be tested, such as*

$$u_1 + u_2 + u_3 + u_4 + u_5 + \cdots$$

from the **k***th term (that is, any term) on is equal to or greater than the corresponding term of a known divergent series of positive terms, such as*

$$v_1 + v_2 + v_3 + v_4 + v_5 + \cdots,$$

then the given **u** *series is divergent.*

Proof: If

$$u_k \geqq v_k,$$
$$u_{k+1} \geqq v_{k+1},$$
$$u_{k+2} \geqq v_{k+2},$$

and so on, and finally

$$u_n \geqq v_n,$$

then it follows that the sum

$$U_n = u_k + u_{k+1} - u_{k+2} + \cdots + u_n \tag{6}$$

is equal to or greater than the sum

$$V_n = v_k + v_{k+1} + v_{k+2} + \cdots + v_n \tag{7}$$

Since the v series is known to diverge, then by definition the sum V_n increases without bound as n increases indefinitely. Since $U_n \geqq V_n$, the sum U_n also increases without bound as n increases indefinitely.

The sum of the first $k - 1$ terms of the series to be tested is

$$S_{k-1} = u_1 + u_2 + u_3 + \cdots + u_{k-1}. \tag{8}$$

Since this is the sum of a finite number of terms, the sum S_{k-1} must be finite.

Since

$$S_n = S_{k-1} + U_n, \tag{9}$$

then

$$\lim_{n \to \infty} S_n = S_{k-1} + \lim_{n \to \infty} U_n. \tag{10}$$

It follows from this that S_n grows larger without bound as n increases without bound, since it was shown that U_n increases without bound as n increases without bound. Therefore the given infinite series

$$u_1 + u_2 + u_3 + u_4 + u_5 + \cdots$$

is a divergent series.

It was shown in the example given in Article 104 that any infinite series whose terms form an infinite geometric progression

with ratio less than 1 is a convergent series whose sum is $\frac{a}{1-r}$, and any infinite series whose terms form an infinite geometric progression with ratio greater than 1 is a divergent series. Hence, infinite series whose terms form a geometric progression are very desirable types of series to use for the comparison tests.

Example 1. Test whether the series

$$1 + \frac{1}{2} + \frac{1}{3^2} + \frac{1}{4^3} + \cdots + \frac{1}{n^{n-1}} + \cdots$$

is a convergent series or not.

Solution. Let us compare this series with the known convergent series whose terms form a geometric progression with ratio $\frac{1}{2}$:

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} + \cdots.$$

By comparing the nth terms of the two series, it is necessary to determine whether or not

$$\frac{1}{n^{n-1}} \leqq \frac{1}{2^{n-1}}.$$

Extracting the $(n-1)$st root of both members, we have

$$\frac{1}{n} \leqq \frac{1}{2},$$

which is a true inequality for $n \geqq 2$. Therefore, each term of the given series after the second term is less than the corresponding term of a known convergent series. Hence, the given series is a convergent series.

Example 2. Prove that

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \cdots$$

is a divergent series.

Solution. This series can be shown to be divergent by the following grouping of terms

$$1 + (\tfrac{1}{2}) + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + (\tfrac{1}{9} + \cdots + \tfrac{1}{16}) + \cdots.$$

The sum of the first n terms of this series is evidently greater than the sum of the first n terms of

$$1 + \tfrac{1}{2} \quad + \tfrac{1}{2} \qquad + \tfrac{1}{2} \qquad\qquad + \tfrac{1}{2} \qquad\qquad + \cdots,$$

since the sum in each pair of parentheses is greater than $\frac{1}{2}$. It is evident for example that

$$\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} > \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{4}{8} = \tfrac{1}{2}$$

since $\frac{1}{5} > \frac{1}{8}$, $\frac{1}{6} > \frac{1}{8}$, and $\frac{1}{7} > \frac{1}{8}$.

Therefore, it is possible to find enough terms in the series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \cdots$$

to make the sum greater than any number of one-halves we please; thus the series is divergent. This is a very slowly divergent series and one that is commonly used for comparison purposes.

Example 3. Prove that

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots + \frac{1}{\sqrt{n}} + \cdots$$

is a divergent series.

Solution. Let us compare this series with the divergent series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \cdots .$$

Comparing their nth terms, it is necessary to determine whether or not

$$\frac{1}{\sqrt{n}} \geqq \frac{1}{n}.$$

Squaring both members gives

$$\frac{1}{n} \geqq \frac{1}{n^2};$$

multiplying by n^2 gives

$$n \geqq 1.$$

Therefore, each term of the first series after the first term is greater than the corresponding term of the known divergent series. Thus the given series is a divergent series.

Another useful series for comparison purposes is the following.

Example 4. The series, commonly known as the p series,

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \frac{1}{5^p} + \cdots$$

is convergent if $p > 1$, and divergent if $p \leqq 1$.

Solution. To prove this series converges for $p > 1$, let us consider the following grouping:

$$1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \cdots + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \cdots + \frac{1}{15^p}\right) + \cdots .$$

For $p > 1$, the sum of this series is evidently less than the sum of the series

$$1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots,$$

which is a convergent series for $p > 1$ since its terms form an infinite geometric progression with ratio less than 1. The ratio $\frac{2}{2^p}$ is less than 1 when $p > 1$. It is evident for example that

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} = \frac{4}{4^p},$$

since $\frac{1}{5^p} < \frac{1}{4^p}$, $\frac{1}{6^p} < \frac{1}{4^p}$, and $\frac{1}{7^p} < \frac{1}{4^p}$.
Therefore the p series converges if $p > 1$.

To prove that the p series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots$$

diverges if $p < 1$, let us compare this series with the known divergent series (see Example 2)

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots.$$

Comparing their nth terms,

$$\frac{1}{n^p} \geqq \frac{1}{n};$$

multiplying by n^p gives

$$1 \geqq n^{p-1}$$

which is true for $n > 1$ and $p < 1$.

Thus we have shown that the p series is convergent if $p > 1$, and divergent if $p \leqq 1$.

In Example 2 we proved that the p series diverges for $p = 1$.

THEOREM III. *For a series to be convergent it is necessary that the* ***n****th term approach zero as a limit as* ***n*** *increases without bound; that is,* $\lim_{n \to \infty} u_n = 0$. The converse of this theorem is not necessarily true; that is, the limit of the nth term can be zero without the series being convergent.

ILLUSTRATION 1. In the convergent series (see Example 1)

$$1 + \frac{1}{2} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^4} + \cdots + \frac{1}{n^{n-1}} + \cdots,$$

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n^{n-1}} = 0.$$

ILLUSTRATION 2. By Example 2, the following series diverges

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{n} + \cdots;$$

however,

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n} = 0.$$

THEOREM IV. *If the* ***n****th term of a series does not approach zero as a limit as* ***n*** *increases without bound, then the series is divergent.*

ILLUSTRATION 3. The series

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \cdots + \frac{n}{n+1} + \cdots$$

is divergent, because

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{n}{n+1} = 1.$$

An alternating series is a series whose terms are alternately positive and negative.

THEOREM V. *A series whose terms are alternately positive and negative is convergent if the numerical value of each term is less than the preceding term, and if the* ***n****th term approaches zero as a limit when* ***n*** *increases without bound.*

ILLUSTRATION 4. The alternating series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + (-1)^{n+1}\frac{1}{n} + (-1)^{n+2}\frac{1}{n+1} + \cdots$$

is convergent since $\frac{1}{n+1} < \frac{1}{n}$ for all values of n, and $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n} = 0$.

THEOREM VI. *A series whose terms are not all of the same sign is convergent if the series whose terms are the absolute values of its terms is convergent.*

By **absolute value** we mean the numerical value without regard to sign; that is, the absolute value of -5 is 5, and we express the absolute value in the following manner, $|-5| = 5$.

A series is said to be **absolutely convergent** if the series of the absolute values of its terms is convergent.

ILLUSTRATION 5. The series

$$1 - \frac{1}{2} + \frac{1}{3^2} - \frac{1}{4^3} + \frac{1}{5^4} - \cdots$$

is absolutely convergent, since the series composed of the absolute values

$$1 + \frac{1}{2} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^4} + \cdots$$

is convergent by Example 1.

If a series with positive and negative signs is convergent but the series whose terms are the absolute values of its terms is divergent, then the series is said to be **conditionally convergent.**

ILLUSTRATION 6. The series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots$$

is conditionally convergent, for by Illustration 4 this series converges, but the corresponding series of positive terms is divergent by Example 2.

106. RATIO TEST

THEOREM. *If in the infinite series of positive terms*

$$u_1 + u_2 + u_3 + u_4 + \cdots + u_n + u_{n+1} + \cdots$$

the ratio $\frac{u_{n+1}}{u_n}$ [*the ratio of the* $(n+1)$st *term to the* **n***th term*], *approaches a limit* $\boldsymbol{R}$ *as* $\boldsymbol{n}$ *increases without bound, then the series is convergent if* $\boldsymbol{R < 1}$ *and divergent if* $\boldsymbol{R > 1}$. *If this ratio is less than* **1** *but approaches* **1** *as* $\boldsymbol{n}$ *increases indefinitely, the test fails to give us any information. If the ratio from some term on is equal to or greater than* **1** *the series evidently diverges.*

Proof: If $\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = R > 1$, the theorem is evident without proof.

If $\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = R < 1$, it follows by the definition of a limit that there exists some value for k such that for all terms after and including the kth term, the ratio $\frac{U_{n+1}}{U_n}$, for $n \geqq k$, is less than r, where $R < r < 1$. That is,

$$\frac{u_{k+1}}{u_k} < r \quad \text{or} \quad u_{k+1} < ru_k,$$

$$\frac{u_{k+2}}{u_{k+1}} < r \quad \text{or} \quad u_{k+2} < ru_{k+1} < r(ru_k) = r^2u_k,$$

$$\frac{u_{k+3}}{u_{k+2}} < r \quad \text{or} \quad u_{k+3} < ru_{k+2} < r^3u_k,$$

and so on. Hence,

$$u_{k+1} + u_{k+2} + u_{k+3} + \cdots < u_k(r + r^2 + r^3 + \cdots) \quad (11)$$

$$< u_k \frac{r}{1 - r}, \quad (12)$$

since the terms in the right-hand member of Equation (11) form an infinite geometric series with ratio r which is less than 1. Therefore, the given infinite series

$$u_1 + u_2 + u_3 + u_4 + \cdots + u_k + u_{k+1} + \cdots$$

converges because the sum of the first k terms is finite, and we have proved that the sum of the terms after the kth term is less than the number $u_k \frac{r}{1 - r}$.

In the cases $\lim_{n \to \infty} \frac{U_{n+1}}{U_n} = 1$, it is necessary to use other tests to determine whether the series is convergent or divergent.

From Theorem VI, Article 105, *the ratio test might be expressed as follows: In the infinite series*

$$u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} + \cdots,$$

the series converges, if

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| < 1;$$

the series diverges, if

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| > 1;$$

the test fails to give us any information if

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = 1.$$

Example 1. Test the following series for convergence or divergence.

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots + \frac{n}{2^n} + \frac{n+1}{2^{n+1}} + \cdots.$$

Solution. The ratio of the $(n+1)$st term to the nth term is

$$\frac{u_{n+1}}{u_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2}\frac{(n+1)}{n}.$$

Then

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{2}\frac{(n+1)}{n} = \frac{1}{2},$$

since $\lim_{n\to\infty} \frac{n+1}{n} = 1$. Hence the given infinite series is convergent.

Example 2. Test the following series for convergence or divergence.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \frac{1}{1+n} + \cdots.$$

Solution. The ratio of the $(n+1)$st term to the nth term is

$$\frac{u_{n+1}}{u_n} = \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1}.$$

Then

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Since this limit is 1, the test fails to give us any information. However, in Example 2, Article 105, this series was shown to be divergent.

107. POWER SERIES

A series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{n-1}x^{n-1} + \cdots,$$

in which the a's, the coefficients, are constants, is called a **power series** in x.

The general ratio test may be used to determine the interval of convergence, or of divergence. However, the ratio test may fail to give any information for the values at the two ends of the interval. It is necessary to use other tests to test for convergence or divergence for those values of x for which $\lim_{x\to\infty} \frac{u_{n+1}}{u_n} = 1$.

Example. Find the interval of convergence and divergence of the series

$$1 + x + \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} + \cdots + \frac{x^{n-1}}{\sqrt{n-1}} + \frac{x^n}{\sqrt{n}} + \cdots.$$

Solution.

$$R = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^n}{\sqrt{n}} \cdot \frac{\sqrt{n-1}}{x^{n-1}} = \lim_{n\to\infty} \frac{\sqrt{n-1}}{\sqrt{n}}\, x = x.$$

It follows from the ratio test theorem, Article 106, that this series converges for all values of x for which $|x| < 1$, and diverges for all values of x for which $|x| > 1$. That is, the series converges for $-1 < x < 1$, and diverges for $x > 1$ and $x < -1$. The ratio test fails to give us any information for $|x| = 1$.

By Example 3, Article 105, this series diverges for $x = 1$, and by Theorem V, Article 105, the series converges for $x = -1$. Therefore, the given series converges for all values of x for which $-1 \leqq x < 1$, and diverges for all other values of x.

EXERCISES

Determine whether the following series (Exercises 1 through 20) are convergent or divergent series.

1. $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$.

2. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$.

3. $1 + \frac{1}{2} + \frac{1}{3^2} + \frac{1}{4^3} + \cdots$.

4. $\frac{1\cdot 2}{3\cdot 5} + \frac{2\cdot 3}{5\cdot 7} + \frac{3\cdot 4}{7\cdot 9} + \frac{4\cdot 5}{9\cdot 11} + \cdots$.

5. $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots$.

6. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \cdots$.

7. $1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \cdots$.

8. $\frac{1}{2} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \cdots$.

9. $1 + \frac{2^2}{2!} + \frac{3^3}{3!} + \frac{4^4}{4!} + \cdots$.

10. $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots$.

11. $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \cdots$.

12. $1 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$.

13. $\frac{3}{1\cdot 2} + \frac{4}{2\cdot 3} + \frac{5}{3\cdot 4} + \frac{6}{4\cdot 5} + \cdots$.

14. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$.

Hint: Each term may be written as follows: $\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots$.

15. $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$.

16. $1 + \frac{2!}{2^2} + \frac{3!}{3^2} + \frac{4!}{4^2} + \cdots$.

17. $1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots$.

18. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \cdots$.

19. $\frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 2^2} + \frac{1}{5 \cdot 2^3} - \frac{1}{7 \cdot 2^4} + \cdots$.

20. $\frac{1}{1} + \frac{1 \cdot 2}{3 \cdot 4} + \frac{1 \cdot 2 \cdot 3}{4 \cdot 5 \cdot 6} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{5 \cdot 6 \cdot 7 \cdot 8} + \cdots$.

For what values of x are the following series (Exercises 21 through 34) convergent?

21. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$.

22. $1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \cdots$.

23. $1 + 4x + 9x^2 + 16x^3 + \cdots$.

24. $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$.

25. $x + \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} + \frac{x^4}{\sqrt{4}} + \cdots$.

26. $\frac{1}{2^2} + \frac{2x}{3^2} + \frac{3x^2}{4^2} + \frac{4x^3}{5^2} + \cdots$.

27. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$.

28. $\frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{6} + \frac{x^4}{8} + \frac{x^5}{10} + \cdots$.

29. $1 + \frac{x}{3} + \frac{x^2}{3^2} + \frac{x^3}{3^3} + \frac{x^4}{3^4} + \cdots$.

30. $\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} + \frac{1}{x+3} + \cdots$.

31. $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \frac{x^4}{4 \cdot 5} + \cdots$.

32. $x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \cdots$.

33. $\frac{1}{2} + \frac{3x}{2^2} + \frac{3^2x^2}{2^3} + \frac{3^3x^3}{2^4} + \cdots$.

34. $\frac{x}{1 \cdot 2} - \frac{x^2}{3 \cdot 2^2} + \frac{x^3}{5 \cdot 2^3} - \frac{x^4}{7 \cdot 2^4} + \cdots$.

From calculus we have the following series (Exercises 35 through 46). Find the interval of convergence for each series.

35. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$.

36. $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$ ($x = 1$ substituted in Exercise 35.)

37. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$.

38. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$.

39. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$.

40. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \cdots$.

41. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$.

42. $\arcsin x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \cdots$.

43. $\frac{\pi}{6} = \frac{1}{2} + \frac{1 \cdot 1 \cdot 1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3 \cdot 1 \cdot 1}{2 \cdot 4 \cdot 5 \cdot 2^5} + \frac{1 \cdot 3 \cdot 5 \cdot 1 \cdot 1}{2 \cdot 4 \cdot 6 \cdot 7 \cdot 2^7} + \cdots$.

$\left(x = \frac{1}{2}$ substituted in Exercise 42.$\right)$

44. $\sqrt{a^2 + x^2} = a + \frac{1}{2} \cdot \frac{x^2}{a} - \frac{1 \cdot 1}{2^2 \cdot 2!} \cdot \frac{x^4}{a^3} + \frac{1 \cdot 3}{2^3 \cdot 3!} \cdot \frac{x^6}{a^5} - \frac{1 \cdot 3 \cdot 5}{2^4 \cdot 4!} \cdot \frac{x^8}{a^7} + \cdots$.

45. $\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$.

46. $\log_e (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots$.

The series for $\log_e (1 + x)$ is convergent for $-1 < x \leqq 1$, and the series for $\log_e (1 - x)$ is convergent for $-1 \leqq x < 1$. By taking the difference of the members of the two series given in Exercises 45 and 46, we have the series shown in Exercise 47.

47. $\log_e (1 + x) - \log_e (1 - x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \cdots\right)$.

This series converges for $x < 1$. Now if we let

$$x = \frac{1}{2N + 1},$$

then x will be less than 1 for all positive values of N, and

$$\frac{1+x}{1-x} = \frac{N+1}{N}.$$

Then by substituting, we have

$$\log_e (1+x) - \log_e (1-x) = \log_e \frac{1+x}{1-x} = \log_e (N+1) - \log_e N,$$

whence we may write the series shown in Exercise 48.

48. $\log_e (N+1) = \log_e N +$

$$2\left[\frac{1}{2N+1} + \frac{1}{3(2N+1)^3} + \frac{1}{5(2N+1)^5} + \cdots\right]$$

This series converges for all positive values of N, and therefore it can be used to compute the natural logarithm of any positive number, provided we know the natural logarithm of the next smaller number. Since $\log_e 1 = 0$, it is thus possible to compute the natural logarithm of any positive integer. As an exercise, the student should show how the natural logarithms of decimal fractions can be calculated from the natural logarithms of positive integers.

TABLE I. Powers, Roots, and Reciprocals of Numbers 1 to 100

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
1	1	1	1.000	1.000	1.0000
2	4	8	1.414	1.260	.5000
3	9	27	1.732	1.442	.3333
4	16	64	2.000	1.587	.2500
5	25	125	2.236	1.710	.2000
6	36	216	2.449	1.817	.1667
7	49	343	2.646	1.913	.1429
8	64	512	2.828	2.000	.1250
9	81	729	3.000	2.080	.1111
10	100	1,000	3.162	2.154	.1000
11	121	1,331	3.317	2.224	.0909
12	144	1,728	3.464	2.289	.0833
13	169	2,197	3.606	2.351	.0769
14	196	2,744	3.742	2.410	.0714
15	225	3,375	3.873	2.466	.0667
16	256	4,096	4.000	2.520	.0625
17	289	4,913	4.123	2.571	.0588
18	324	5,832	4.243	2.621	.0556
19	361	6,859	4.359	2.668	.0526
20	400	8,000	4.472	2.714	.0500
21	441	9,261	4.583	2.759	.0476
22	484	10,648	4.690	2.802	.0455
23	529	12,167	4.796	2.844	.0435
24	576	13,824	4.899	2.884	.0417
25	625	15,625	5.000	2.924	.0400
26	676	17,576	5.099	2.962	.0385
27	729	19,683	5.196	3.000	.0370
28	784	21,952	5.292	3.037	.0357
29	841	24,389	5.385	3.072	.0345
30	900	27,000	5.477	3.107	.0333
31	961	29,791	5.568	3.141	.0323
32	1,024	32,768	5.657	3.175	.0312
33	1,089	35,937	5.745	3.208	.0303
34	1,156	39,304	5.831	3.240	.0294
35	1,225	42,875	5.916	3.271	.0286
36	1,296	46,656	6.000	3.302	.0278
37	1,369	50,653	6.083	3.332	.0270
38	1,444	54,872	6.164	3.362	.0263
39	1,521	59,319	6.245	3.391	.0256
40	1,600	64,000	6.325	3.420	.0250
41	1,681	68,921	6.403	3.448	.0244
42	1,764	74,088	6.481	3.476	.0238
43	1,849	79,507	6.557	3.503	.0233
44	1,936	85,184	6.633	3.530	.0227
45	2,025	91,125	6.708	3.557	.0222
46	2,116	97,336	6.782	3.583	.0217
47	2,209	103,823	6.856	3.609	.0213
48	2,304	110,592	6.928	3.634	.0208
49	2,401	117,649	7.000	3.659	.0204
50	2,500	125,000	7.071	3.684	.0200
51	2,601	132,651	7.141	3.708	.0196
52	2,704	140,608	7.211	3.733	.0192
53	2,809	148,877	7.280	3.756	.0189
54	2,916	157,464	7.348	3.780	.0185
55	3,025	166,375	7.416	3.803	.0182
56	3,136	175,616	7.483	3.826	.0179
57	3,249	185,193	7.550	3.849	.0175
58	3,364	195,112	7.616	3.871	.0172
59	3,481	205,379	7.681	3.893	.0169
60	3,600	216,000	7.746	3.915	.0167
61	3,721	226,981	7.810	3.936	.0164
62	3,844	238,328	7.874	3.958	.0161
63	3,969	250,047	7.937	3.979	.0159
64	4,096	262,144	8.000	4.000	.0156
65	4,225	274,625	8.062	4.021	.0154
66	4,356	287,496	8.124	4.041	.0152
67	4,489	300,763	8.185	4.062	.0149
68	4,624	314,432	8.246	4.082	.0147
69	4,761	328,509	8.307	4.102	.0145
70	4,900	343,000	8.367	4.121	.0143
71	5,041	357,911	8.426	4.141	.0141
72	5,184	373,248	8.485	4.160	.0139
73	5,329	389,017	8.544	4.179	.0137
74	5,476	405,224	8.602	4.198	.0135
75	5,625	421,875	8.660	4.217	.0133
76	5,776	438,976	8.718	4.236	.0132
77	5,929	456,533	8.775	4.254	.0130
78	6,084	474,552	8.832	4.273	.0128
79	6,241	493,039	8.888	4.291	.0127
80	6,400	512,000	8.944	4.309	.0125
81	6,561	531,441	9.000	4.327	.0123
82	6,724	551,368	9.055	4.344	.0122
83	6,889	571,787	9.110	4.362	.0120
84	7,056	592,704	9.165	4.380	.0119
85	7,225	614,125	9.220	4.397	.0118
86	7,396	636,056	9.274	4.414	.0116
87	7,569	658,503	9.327	4.431	.0115
88	7,744	681,472	9.381	4.448	.0114
89	7,921	704,969	9.434	4.465	.0112
90	8,100	729,000	9.487	4.481	.0111
91	8,281	753,571	9.539	4.498	.0110
92	8,464	778,688	9.592	4.514	.0109
93	8,649	804,357	9.644	4.531	.0108
94	8,836	830,584	9.695	4.547	.0106
95	9,025	857,375	9.747	4.563	.0105
96	9,216	884,736	9.798	4.579	.0104
97	9,409	912,673	9.849	4.595	.0103
98	9,604	941,192	9.899	4.610	.0102
99	9,801	970,299	9.950	4.626	.0101
100	10,000	1,000,000	10.000	4.642	.0100

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n	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

TABLE II. Four-Place Logarithms. (*Continued*)

n	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

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TABLE III. Natural Trigonometric Functions

Angle	Sine	Cosine	Tangent	Cotangent	
0°	.00000	1.00000	.00000	∞	90°
1°	.01745	.99985	.01746	57.2900	89°
2°	.03490	.99939	.03492	28.6363	88°
3°	.05234	.99863	.05241	19.0811	87°
4°	.06976	.99756	.06993	14.3007	86°
5°	.08716	.99619	.08749	11.4301	85°
6°	.10453	.99452	.10510	9.51436	84°
7°	.12187	.99255	.12278	8.14435	83°
8°	.13917	.99027	.14054	7.11537	82°
9°	.15643	.98769	.15838	6.31375	81°
10°	.17365	.98481	.17633	5.67128	80°
11°	.19081	.98163	.19438	5.14455	79°
12°	.20791	.97815	.21256	4.70463	78°
13°	.22495	.97437	.23087	4.33148	77°
14°	.24192	.97030	.24933	4.01078	76°
15°	.25882	.96593	.26795	3.73205	75°
16°	.27564	.96126	.28675	3.48741	74°
17°	.29237	.95630	.30573	3.27085	73°
18°	.30902	.95106	.32492	3.07768	72°
19°	.32557	.94552	.34433	2.90421	71°
20°	.34202	.93969	.36397	2.74748	70°
21°	.35837	.93358	.38386	2.60509	69°
22°	.37461	.92718	.40403	2.47509	68°
23°	.39073	.92050	.42447	2.35585	67°
24°	.40674	.91355	.44523	2.24604	66°
25°	.42262	.90631	.46631	2.14451	65°
26°	.43837	.89879	.48773	2.05030	64°
27°	.45399	.89101	.50953	1.96261	63°
28°	.46947	.88295	.53171	1.88073	62°
29°	.48481	.87462	.55431	1.80405	61°
30°	.50000	.86603	.57735	1.73205	60°
31°	.51504	.85717	.60086	1.66428	59°
32°	.52992	.84805	.62487	1.60033	58°
33°	.54464	.83867	.64941	1.53986	57°
34°	.55919	.82904	.67451	1.48256	56°
35°	.57358	.81915	.70021	1.42815	55°
36°	.58779	.80902	.72654	1.37638	54°
37°	.60182	.79864	.75355	1.32704	53°
38°	.61566	.78801	.78129	1.27994	52°
39°	.62932	.77715	.80978	1.23490	51°
40°	.64279	.76604	.83910	1.19175	50°
41°	.65606	.75471	.86929	1.15037	49°
42°	.66913	.74314	.90040	1.11061	48°
43°	.68200	.73135	.93252	1.07237	47°
44°	.69466	.71934	.96569	1.03553	46°
45°	.70711	.70711	1.00000	1.00000	45°
	Cosine	Sine	Cotangent	Tangent	Angle

TABLE IV. American Experience Table of Mortality

Age x	Number Living l_c	Number Dying d_x	Yearly Probability of Dying q_x	Yearly Probability of Living p_x	Age x	Number Living l_x	Number Dying d_x	Yearly Probability of Dying q_x	Yearly Probability of Living p_x
10	100 000	749	0·007 490	0·992 510	53	66 797	1 091	0·016 333	0·983 667
11	99 251	746	0·007 516	0·992 484	54	65 706	1 143	0·017 396	0·982 604
12	98 505	743	0·007 543	0·992 457	55	64 563	1 199	0·018 571	0·981 429
13	97 762	740	0·007 569	0·992 421	56	63 364	1 260	0·019 885	0·980 115
14	97 022	737	0·007 596	0·992 404	57	62 104	1 325	0·021 335	0·978 665
15	96 285	735	0·007 634	0·992 366	58	60 779	1 394	0·022 936	0·977 064
16	95 550	732	0·007 661	0·992 339	59	59 385	1 468	0·024 720	0·975 280
17	94 818	729	0·007 688	0·992 312	60	57 917	1 546	0·026 693	0·973 307
18	94 089	727	0·007 727	0·992 273	61	56 371	1 628	0·028 880	0·971 120
19	93 362	725	0·007 765	0·992 235	62	54 743	1 713	0·031 292	0·968 708
20	92 637	723	0·007 805	0·992 195	63	53 030	1 800	0·033 943	0·966 057
21	91 914	722	0·007 855	0·992 145	64	51 230	1 889	0·036 873	0·963 127
22	91 192	721	0·007 906	0·992 094	65	49 341	1 980	0·040 129	0·959 871
23	90 471	720	0·007 958	0·992 042	66	47 361	2 070	0·043 707	0·956 293
24	89 751	719	0·008 011	0·991 989	67	45 291	2 158	0·047 647	0·952 353
25	89 032	718	0·008 065	0·991 935	68	43 133	2 243	0·052 002	0·947 998
26	88 314	718	0·008 130	0·991 870	69	40 890	2 321	0·056 762	0·943 238
27	87 596	718	0·008 197	0·991 803	70	38 569	2 391	0·061 993	0·938 007
28	86 878	718	0·008 264	0·991 736	71	36 178	2 448	0·067 665	0·932 335
29	86 160	719	0·008 345	0·991 655	72	33 730	2 487	0·073 733	0·926 267
30	85 441	720	0·008 427	0·991 573	73	31 243	2 505	0·080 178	0·919 282
31	84 721	721	0·008 510	0·991 490	74	28 738	2 501	0·087 028	0·912 972
32	84 000	723	0·008 607	0·991 393	75	26 237	2 476	0·094 371	0·905 629
33	83 277	726	0·008 718	0·991 282	76	23 761	2 431	0·102 311	0·897 689
34	82 551	729	0·008 831	0·991 169	77	21 330	2 369	0·111 064	0·888 936
35	81 822	732	0·008 946	0·991 054	78	18 961	2 291	0·120 827	0·879 173
36	81 090	737	0·009 089	0·990 911	79	16 670	2 196	0·131 734	0·868 266
37	80 353	742	0·009 234	0·990 766	80	14 474	2 091	0·144 466	0·855 534
38	79 611	749	0·009 408	0·990 592	81	12 383	1 964	0·158 605	0·841 395
39	78 862	756	0·009 586	0·990 414	82	10 419	1 816	0·174 297	0·825 703
40	78 106	765	0·009 794	0·990 206	83	8 603	1 648	0·191 561	0·808 439
41	77 341	774	0·010 008	0·989 992	84	6 955	1 470	0·211 359	0·788 641
42	76 567	785	0·010 252	0·989 748	85	5 485	1 292	0·235 552	0·764 448
43	75 782	797	0·010 517	0·989 483	86	4 193	1 114	0·265 681	0·734 319
44	74 985	812	0·010 829	0·989 171	87	3 079	933	0·303 020	0·696 980
45	74 173	828	0·011 163	0·988 837	88	2 146	744	0·346 692	0·653 308
46	73 345	848	0·011 562	0·988 438	89	1 402	555	0·395 863	0·604 137
47	72 497	870	0·012 000	0·988 000	90	847	385	0·454 545	0·545 455
48	71 627	896	0·012 509	0·987 491	91	462	246	0·532 466	0·467 534
49	70 731	927	0·013 106	0·986 894	92	216	137	0·634 259	0·365 741
50	69 804	962	0·013 781	0·986 219	93	79	58	0·734 177	0·265 823
51	68 842	1 001	0·014 541	0·985 459	94	21	18	0·857 143	0·142 857
52	67 841	1 044	0·015 389	0·984 611	95	3	3	1·000 000	0·000 000

TABLE V. Amount of 1 at Compound Interest: $s = (1 + i)^n$

n	1 per cent	1½ per cent	2 per cent	2½ per cent	3 per cent	n
1	1·010 0000	1·015 0000	1·020 0000	1·025 0000	1·030 0000	1
2	1·020 1000	1·030 2250	1·040 4000	1·050 6250	1·060 9000	2
3	1·030 3010	1·045 6784	1·061 2080	1·076 8906	1·092 7270	3
4	1·040 6040	1·061 3636	1·082 4322	1·103 8129	1·125 5088	4
5	1·051 0101	1·077 2841	1·104 0808	1·131 4082	1·159 2741	5
6	1·061 5202	1·093 4433	1·126 1624	1·159 6934	1·194 0523	6
7	1·072 1354	1·109 8449	1·148 6856	1·188 6857	1·229 8739	7
8	1·082 8567	1·126 4926	1·171 6594	1·218 4029	1·266 7701	8
9	1·093 6853	1·143 3900	1·195 0926	1·248 8630	1·304 7732	9
10	1·104 6221	1·160 5408	1·218 9944	1·280 0845	1·343 9164	10
11	1·115 6684	1·177 9489	1·243 3743	1·312 0867	1·384 2339	11
12	1·126 8250	1·195 6182	1·268 2418	1·344 8888	1·425 7609	12
13	1·138 0933	1·213 5524	1·293 6066	1·378 5110	1·468 5337	13
14	1·149 4742	1·231 7557	1·319 4788	1·412 9738	1·512 5897	14
15	1·160 9690	1·250 2321	1·345 8683	1·448 2982	1·557 9674	15
16	1·172 5786	1·268 9856	1·372 7857	1·484 5056	1·604 7064	16
17	1·184 3044	1·288 0203	1·400 2414	1·521 6183	1·652 8476	17
18	1·196 1475	1·307 3406	1·428 2462	1·559 6587	1·702 4330	18
19	1·208 1090	1·326 9507	1·456 8112	1·598 6502	1·753 5060	19
20	1·220 1900	1·346 8550	1·485 9474	1·638 6164	1·806 1112	20
21	1·232 3919	1·367 0578	1·515 6663	1·679 5819	1·860 2946	21
22	1·244 7159	1·387 5637	1·545 9795	1·721 5714	1·916 1034	22
23	1·257 1630	1·408 3772	1·576 8993	1·764 6107	1·973 5865	23
24	1·269 7347	1·429 5028	1·608 4372	1·808 7620	2·032 7941	24
25	1·282 4320	1·450 9457	1·640 6060	1·853 9441	2·093 7779	25
26	1·295 2563	1·472 7095	1·673 4181	1·900 2927	2·156 5913	26
27	1·308 2089	1·494 8002	1·706 8865	1·947 8000	2·221 2890	27
28	1·321 2910	1·517 2222	1·741 0242	1·996 4950	2·287 9277	28
29	1·334 5039	1·539 9805	1·775 8447	2·046 4074	2·356 5655	29
30	1·347 8489	1·563 0802	1·811 3616	2·097 5676	2·427 2625	30
31	1·361 3274	1·586 5264	1·847 5888	2·150 0068	2·500 0804	31
32	1·374 9407	1·610 3243	1·884 5406	2·203 7569	2·575 0828	32
33	1·388 6901	1·634 4792	1·922 2314	2·258 8509	2·652 3352	33
34	1·402 5770	1·658 9964	1·960 6760	2·315 3221	2·731 9052	34
35	1·416 6028	1·683 8813	1·999 8896	2·373 2052	2·813 8625	35
36	1·430 7688	1·709 1395	2·039 8873	2·432 5353	2·898 2783	36
37	1·445 0765	1·734 7766	2·080 6851	2·493 3487	2·985 2267	37
38	1·459 5272	1·760 7983	2·122 2988	2·555 6824	3·074 7835	38
39	1·474 1225	1·787 2102	2·164 7448	2·619 5745	3·167 0270	39
40	1·488 8637	1·814 0184	2·208 0397	2·685 0638	3·262 0378	40
41	1·503 7524	1·841 2287	2·252 2005	2·752 1904	3·359 8989	41
42	1·518 7899	1·868 8471	2·297 2445	2·820 9952	3·460 6959	42
43	1·533 9778	1·896 8798	2·343 1891	2·891 5201	3·564 5168	43
44	1·549 3176	1·925 3330	2·390 0531	2·963 8081	3·671 4523	44
45	1·564 8108	1·954 2130	2·437 8542	3·037 9033	3·781 5958	45
46	1·580 4589	1·983 5267	2.486 6113	3 113 8509	3·895 0437	46
47	1·596 2634	2·013 2791	2·536 3435	3·191 6975	4·011 8950	47
48	1·612 2261	2·043 4783	2·587 0704	3·271 4896	4·132 2519	48
49	1·628 3483	2·074 1305	2·638 8118	3·353 2768	4·256 2194	49
50	1·644 6318	2·105 2424	2·691 5880	3·437 1087	4·383 9061	50
60	1·816 6967	2·443 2198	3·281 0308	4·399 7898	5·891 6031	60
70	2·006 7634	2·835 4563	3·999 5582	5·632 1029	7·917 8219	70
80	2·216 7152	3·290 6628	4·875 4392	7·209 5678	10·640 8906	80
90	2·448 6327	3·818 9485	5·943 1331	9·228 8563	14·300 4671	90
100	2·704 8138	4·432 0456	7·244 6461	11·813 7164	19·218 6320	100

TABLE V. Amount of 1 at Compound Interest: $s = (1 + i)^n$. *(Continued)*

n	$3\frac{1}{2}$ per cent	4 per cent	$4\frac{1}{2}$ per cent	5 per cent	6 per cent	n
1	1·035 0000	1·040 0000	1·045 0000	1·050 0000	1·060 0000	1
2	1·071 2250	1·081 6000	1·092 0250	1·102 5000	1·123 6000	2
3	1·108 7179	1·124 8640	1·141 1661	1·157 6250	1·191 0160	3
4	1·147 5230	1·169 8586	1·192 5186	1·215 5063	1·262 4770	4
5	1·187 6863	1·216 6529	1·246 1819	1·276 2816	1·338 2256	5
6	1·229 2553	1·265 3190	1·302 2601	1·340 0956	1·418 5191	6
7	1·272 2792	1·315 9308	1·360 8618	1·407 1004	1·503 6302	7
8	1·316 8090	1·368 5691	1·422 1006	1·477 4554	1·593 8481	8
9	1·362 8794	1·423 3118	1·486 0951	1·551 3282	1·689 4790	9
10	1·410 5988	1·480 2443	1·552 9694	1·628 8946	1·790 8477	10
11	1·459 9697	1·539 4541	1·622 8530	1·710 3394	1·898 2986	11
12	1·511 0687	1·601 0322	1·695 8814	1·795 8563	2·012 1965	12
13	1·563 9561	1·665 0735	1·772 1961	1·885 6491	2·132 9283	13
14	1·618 6945	1·731 6764	1·851 9449	1·979 9316	2·260 9040	14
15	1·675 3488	1·800 9435	1·935 2824	2·078 9282	2·396 5582	15
16	1·733 9860	1·872 9812	2·022 3701	2·182 8746	2·540 3517	16
17	1·794 6756	1·947 9005	2·113 3768	2·292 0183	2·692 7728	17
18	1·857 4892	2·025 8165	2·208 4788	2·406 6192	2·854 3392	18
19	1·922 5013	2·106 8492	2·307 8603	2·526 9502	3·025 5995	19
20	1·989 7889	2·191 1231	2·411 7140	2·653 2977	3·207 1355	20
21	2·059 4315	2·278 7681	2·520 2412	2·785 9626	3·399 5636	21
22	2·131 5116	2·369 9188	2·633 6520	2·925 2608	3·603 5374	22
23	2·206 1145	2·464 7155	2·752 1664	3·071 5238	3·819 7497	23
24	2·283 3285	2·563 3042	2·876 0138	3·225 0999	4·048 9346	24
25	2·363 2450	2·665 8363	3·005 4345	3·386 3549	4·291 8706	25
26	2·445 9586	2·772 4698	3·140 6790	3·555 6727	4·549 3830	26
27	2·531 5671	2·883 3686	3·282 0096	3·733 4563	4·822 3459	27
28	2·620 1720	2·998 7033	3·429 7000	3·920 1291	5·111 6867	28
29	2·711 8780	3·118 6515	3·584 0365	4·116 1356	5·418 3879	29
30	2·806 7934	3·243 3975	3·745 3181	4·321 9424	5·743 4912	30
31	2·905 0315	3·373 1334	3·913 8575	4·538 0395	6·088 1006	31
32	3·006 7076	3·508 0588	4·089 9810	4·764 9415	6·453 3867	32
33	3·111 9424	3·648 3811	4·274 0302	5·003 1885	6·840 5899	33
34	3·220 8603	3·794 3163	4·466 3615	5·253 3480	7·251 0253	34
35	3·333 5904	3·946 0890	4·667 3473	5·516 0154	7·686 0868	35
36	3·450 2661	4·103 9326	4·877 3785	5·791 8161	8·147 2520	36
37	3·571 0254	4·268 0899	5·096 8605	6·081 4069	8·636 0871	37
38	3·696 0113	4·438 8135	5·326 2192	6·385 4773	9·154 2524	38
39	3·825 3717	4·616 3660	5·565 8991	6·704 7512	9·703 5075	39
40	3·959 2597	4·801 0206	5·816 3645	7·039 9887	10·285 7179	40
41	4·097 8338	4·993 0615	6·078 1009	7·391 9882	10·902 8610	41
42	4·241 2580	5·192 7839	6·351 6155	7·761 5876	11·557 0327	42
43	4·389 7020	5·400 4953	6·637 4382	8·149 6669	12·250 4546	43
44	4·543 3416	5·616 5151	6·936 1229	8·557 1503	12·985 4819	44
45	4·702 3586	5·841 1757	7·248 2484	8·985 0078	13·764 6104	45
46	4·866 9411	6·074 8237	7·574 4196	9·434 2586	14·590 4875	46
47	5·037 2840	6·317 8156	7·915 2685	9·905 9711	15·465 9167	47
48	5·213 5890	6·570 5382	8·271 4556	10·401 2696	16·393 8717	48
49	5·396 0646	6·833 3494	8·643 6711	10·921 3331	17·377 5040	49
50	5·584 9264	7·106 6834	9·032 6363	11·467 3998	18·420 1543	50
60	7·878 0909	10·519 6274	14·027 4079	18·679 1859	32·987 6908	60
70	11·112 8253	15·571 6184	21·784 1356	30·426 4255	59·075 9302	70
80	15·675 7375	23·049 7991	33·830 0964	49·561 4411	105·795 9935	80
90	22·112 1760	34·119 3333	52·537 1053	80·730 3650	189·464 5112	90
100	31·191 4080	50·504 9482	81·588 5180	131·501 2578	339·302 0835	100

TABLE VI. Present Value of 1 Due in n Periods : $v^n = (1 + i)^{-n}$

n	1 per cent	1½ per cent	2 per cent	2½ per cent	3 per cent	n
1	0·990 0990	0·985 2219	0·980 3922	0·975 6097	0·970 8738	1
2	0·980 2960	0·970 6618	0·961 1688	0·951 8144	0·942 5959	2
3	0·970 5902	0·956 3170	0·942 3223	0·928 5994	0·915 1416	3
4	0·960 9803	0·942 1842	0·923 8454	0·905 9506	0·888 4870	4
5	0·951 4656	0·928 2603	0·905 7306	0·883 8543	0·862 6088	5
6	0·942 0452	0·914 5422	0·887 9714	0·862 2969	0·837 4843	6
7	0·932 7180	0·901 0268	0·870 5602	0·841 2652	0·813 0915	7
8	0·923 4832	0·887 7111	0·853 4904	0·820 7466	0·789 4092	8
9	0·914 3398	0·874 5922	0·836 7553	0·800 7284	0·766 4167	9
10	0·905 2870	0·861 6672	0·820 3483	0·781 1984	0·744 0939	10
11	0·896 3237	0·848 9332	0·804 2630	0·762 1448	0·722 4213	11
12	0·887 4492	0·836 3874	0·788 4932	0·743 5559	0·701 3799	12
13	0·878 6626	0·824 0270	0·773 0325	0·725 4204	0·680 9513	13
14	0·869 9630	0·811 8493	0·757 8750	0·707 7272	0·661 1178	14
15	0·861 3495	0·799 8515	0·743 0147	0·690 4656	0·641 8620	15
16	0·852 8213	0·788 0310	0·728 4458	0·673 6249	0·623 1669	16
17	0·844 3775	0·776 3853	0·714 1626	0·657 1951	0·605 0164	17
18	0·836 0173	0·762 9116	0·700 1594	0·641 1659	0·587 3946	18
19	0·827 7399	0·753 6075	0·686 4308	0·625 5277	0·570 2860	19
20	0·819 5445	0·742 4704	0·672 9713	0·610 2709	0·553 6758	20
21	0·811 4302	0·731 4980	0·659 7758	0·595 3863	0·537 5493	21
22	0·803 3962	0·720 6876	0·646 8390	0·580 8647	0·521 8925	22
23	0·795 4418	0·710 0370	0·634 1559	0·566 6972	0·506 6917	23
24	0·787 5661	0·699 5439	0·621 7216	0·552 8754	0·491 9377	24
25	0·779 7685	0·689 2058	0·609 5309	0·539 3906	0·477 6056	25
26	0·772 0480	0·679 0205	0·597 5793	0·526 2347	0·463 6947	26
27	0·764 4039	0·668 9857	0·585 8620	0·513 3997	0·450 1891	27
28	0·756 8356	0·659 0993	0·574 3746	0·500 8778	0·437 0768	28
29	0·749 3422	0·649 3589	0·563 1123	0·488 6613	0·424 3464	29
30	0·741 9229	0·639 7624	0·552 0709	0·476 7427	0·411 9868	30
31	0·734 5772	0·630 3078	0·541 2460	0·465 1148	0·399 9872	31
32	0·727 3041	0·620 9929	0·530 6333	0·453 7706	0·388 3370	32
33	0·720 1031	0·611 8157	0·520 2287	0·442 7030	0·377 0262	33
34	0·712 9733	0·602 7741	0·510 0282	0·431 9053	0·366 0449	34
35	0·705 9142	0·593 8661	0·500 0276	0·421 3711	0·355 3834	35
36	0·698 9250	0·585 0897	0·490 2232	0·411 0937	0·345 0324	36
37	0·692 0049	0·576 4431	0·480 6109	0·401 0670	0·334 9829	37
38	0·685 1534	0·567 9242	0·471 1872	0·391 2849	0·325 2262	38
39	0·678 3697	0·559 5313	0·461 9482	0·381 7414	0·315 7535	39
40	0·671 6531	0·551 2623	0·452 8904	0·372 4306	0·306 5568	40
41	0·665 0031	0·543 1156	0·444 0102	0·363 3470	0·297 6280	41
42	0·658 4189	0·535 0892	0·435 3041	0·354 4848	0·288 9592	42
43	0·651 8999	0·527 1815	0·426 7688	0·345 8389	0·280 5429	43
44	0·645 4455	0·519 3907	0·418 4007	0·337 4038	0·272 3718	44
45	0·639 0549	0·511 7149	0·410 1968	0·329 1744	0·264 4386	45
46	0·632 7276	0·504 1527	0·402 1537	0·321 1457	0·256 7365	46
47	0·626 4630	0·496 7021	0·394 2684	0·313 3129	0·249 2588	47
48	0·620 2604	0·489 3617	0·386 5377	0·305 6712	0·241 9988	48
49	0·614 1193	0·482 1298	0·378 9584	0·298 2158	0·234 9503	49
50	0·608 0388	0·475 0047	0·371 5279	0·290 9422	0·228 1071	50
60	0·550 4496	0·409 2960	0·304 7823	0·227 2836	0·169 7331	60
70	0·498 3149	0·352 6769	0·250 0276	0·177 5536	0·126 2974	70
80	0·451 1179	0·303 8902	0·205 1097	0·138 7046	0·093 9771	80
90	0·408 3912	0·261 8522	0·168 2614	0·108 3558	0·069 9278	90
100	0·369 7112	0·225 6294	0·138 0330	0·084 6474	0·052 0328	100

TABLE VI. Present Value of 1 Due in n Periods: $v^n = (1 + i)^{-n}$. *(Continued)*

n	3½ per cent	4 per cent	4½ per cent	5 per cent	6 per cent	n
1	0·966 1837	0·961 5385	0·956 9378	0·952 3810	0·943 3962	1
2	0·933 5107	0·924 5562	0·915 7300	0·907 0295	0·889 9964	2
3	0·901 9427	0·888 9964	0·876 2966	0·863 8376	0·839 6192	3
4	0·871 4422	0·854 8042	0·838 5614	0·822 7024	0·792 0937	4
5	0·841 9732	0·821 9271	0·802 4510	0·783 5262	0·747 2582	5
6	0·813 5006	0·790 3145	0·767 8957	0·746 2154	0·704 9605	6
7	0·785 9910	0·759 9177	0·734 8285	0·710 6813	0·665 0571	7
8	0·759 4116	0·730 6902	0·703 1851	0·676 8394	0·627 4124	8
9	0·733 7310	0·702 5867	0·672 9044	0·644 6089	0·591 8985	9
10	0·708 9188	0·675 5642	0·643 9277	0·613 9132	0·558 3948	10
11	0·684 9457	0·649 5809	0·616 1987	0·584 6793	0·526 7875	11
12	0·661 7833	0·624 5970	0·589 6639	0·556 8374	0·496 9694	12
13	0·639 4042	0·600 5741	0·564 2716	0·530 3214	0·468 8390	13
14	0·617 7818	0·577 4751	0·539 9729	0·505 0680	0·442 3010	14
15	0·596 8906	0·555 2645	0·516 7204	0·481 0171	0·417 2651	15
16	0·576 7059	0·533 9082	0·494 4693	0·458 1115	0·393 6463	16
17	0·557 2038	0·513 3732	0·473 1764	0·436 2967	0·371 3644	17
18	0·538 3611	0·493 6281	0·452 8004	0·415 5207	0·350 3438	18
19	0·520 1557	0·474 6424	0·433 3018	0·395 7340	0·330 5130	19
20	0·502 5659	0·456 3870	0·414 6429	0·376 8895	0·311 8047	20
21	0·485 5709	0·438 8336	0·396 7875	0·358 9424	0·294 1554	21
22	0·469 1506	0·421 9554	0·379 7009	0·341 8499	0·277 5051	22
23	0·453 2856	0·405 7263	0·363 3501	0·325 5713	0·261 7972	23
24	0·437 9572	0·390 1215	0·347 7035	0·310 0679	0·246 9786	24
25	0·423 1470	0·375 1168	0·332 7306	0·295 3027	0·232 9986	25
26	0·408 8377	0·360 6893	0·318 4025	0·281 2407	0·219 8100	26
27	0·395 0122	0·346 8166	0·304 6914	0·267 8483	0·207 3680	27
28	0·381 6543	0·333 4775	0·291 5707	0·255 0936	0·195 6301	28
29	0·368 7482	0·320 6514	0·279 0150	0·242 9463	0·184 5567	29
30	0·356 2784	0·308 3186	0·267 0000	0·231 3774	0·174 1101	30
31	0·344 2304	0·296 4603	0·255 5024	0·220 3595	0·164 2548	31
32	0·332 5897	0·285 0579	0·244 4999	0·209 8662	0·154 9574	32
33	0·321 3427	0·274 0942	0·233 9712	0·199 8725	0·146 1862	33
34	0·310 4761	0·263 5521	0·223 8959	0·190 3548	0·137 9115	34
35	0·299 9769	0·253 4155	0·214 2544	0·181 2903	0·130 1052	35
36	0·289 8327	0·243 6687	0·205 0281	0·172 6574	0·122 7408	36
37	0·280 0316	0·234 2968	0·196 1992	0·164 4356	0·115 7932	37
38	0·270 5619	0·225 2854	0·187 7504	0·156 6054	0·109 2388	38
39	0·261 4125	0·216 6206	0·179 6655	0·149 1480	0·103 0555	39
40	0·252 5725	0·208 2890	0·171 9287	0·142 0457	0·097 2222	40
41	0·244 0314	0·200 2779	0·164 5251	0·135 2816	0·091 7190	41
42	0·235 7791	0·192 5749	0·157 4403	0·128 8396	0·086 5274	42
43	0·227 8059	0·185 1682	0·150 6605	0·122 7044	0·081 6296	43
44	0·220 1023	0·178 0464	0·144 1728	0·116 8613	0·077 0091	44
45	0·212 6592	0·171 1984	0·137 9644	0·111 2965	0·072 6501	45
46	0·205 4679	0·164 6139	0·132 0233	0·105 9966	0·068 5378	46
47	0·198 5197	0·158 2826	0·126 3381	0·100 9492	0·064 6583	47
48	0·191 8065	0·152 1948	0·120 8977	0·096 1421	0·060 9984	48
49	0·185 3202	0·146 3411	0·115 6916	0·091 5639	0·057 5457	49
50	0·179 0534	0·140 7126	0·110 7096	0·087 2037	0·054 2884	50
60	0·126 9343	0·095 0604	0·071 2890	0·053 5355	0·030 3143	60
70	0·089 9861	0·064 2194	0·045 9050	0·032 8662	0·016 9274	70
80	0·063 7928	0·043 3843	0·029 5595	0·020 1770	0·009 4522	80
90	0·045 2240	0·029 3089	0·019 0342	0·012 3869	0·005 2780	90
100	0·032 0601	0·019 8000	0·012 2566	0·007 6045	0·002 9472	100

TABLE VII. Amount of Annuity: $s_{\overline{n}|} = \dfrac{(1+i)^{-n}-1}{i}$

n	1 per cent	1½ per cent	2 per cent	2½ per cent	3 per cent	n
1	1·000 0000	1·000 0000	1·000 0000	1·000 0000	1·000 0000	1
2	2·010 0000	2·015 0000	2·020 0000	2·025 0000	2·030 0000	2
3	3·030 1000	3·045 2250	3·060 4000	3·075 6250	3·090 9000	3
4	4·060 4010	4·090 9034	4·121 6080	4·152 5156	4·183 6270	4
5	5·101 0050	5·152 2669	5·204 0402	5·256 3285	5·309 1358	5
6	6·152 0150	6·299 5509	6·308 1210	6·387 7367	6·468 4099	6
7	7·213 5352	7·322 9945	7·434 2834	7·547 4201	7·662 4622	7
8	8·285 6706	8·432 8391	8·582 9690	8·736 1159	8·892 3361	8
9	9·368 5273	9·559 3317	9·754 6283	9·954 5188	10·159 1061	9
10	10·462 2125	10·702 7217	10·949 7210	11·203 3818	11·463 8793	10
11	11·566 8347	11·863 2625	12·168 7154	12·483 4663	12·807 7957	11
12	12·682 5030	13·041 2114	13·412 0897	13·795 5530	14·192 0296	12
13	13·809 3280	14·236 8296	14·680 3315	15·140 4418	15·617 7904	13
14	14·947 4213	15·450 3820	15·973 9382	16·518 9528	17·086 3242	14
15	16·096 8955	16·682 1378	17·293 4169	17·931 9267	18·598 9139	15
16	17·257 8645	17·932 3698	18·639 2853	19·380 2248	20·156 8813	16
17	18·430 4431	19·201 3554	20·012 0710	20·864 7304	21·761 5877	17
18	19·614 7476	20·489 3757	21·412 3124	22·386 3487	23·414 4354	18
19	20·810 8950	21·796 7164	22·840 5586	23·946 0074	25·116 8684	19
20	22·019 0045	23·123 6671	24·297 3698	25·544 6576	26·870 3745	20
21	23·239 1940	24·470 5221	25·783 3172	27·183 2741	28·676 4857	21
22	24·471 5860	25·837 5799	27·298 9835	28·862 8559	30·536 7803	22
23	25·716 3018	27·225 1436	28·844 9632	30·584 4273	32·452 8837	23
24	26·973 4648	28·633 5208	30·421 8625	32·349 0380	34·426 4702	24
25	28·243 1995	30·063 0236	32·030 2997	34·157 7639	36·459 2643	25
26	29·525 6315	31·513 9690	33·670 9057	36·011 7080	38·553 0423	26
27	30·820 8878	32·986 6785	35·344 3238	37·912 0007	40·709 6335	27
28	32·129 0967	34·481 4787	37·051 2103	39·859 8008	42·930 9215	28
29	33·450 3877	35·998 7009	38·792 2345	41·856 2958	45·218 8502	29
30	34·784 8915	37·538 6814	40·568 0792	43·902 7032	47·575 4157	30
31	36·132 7405	39·101 7686	42·379 4408	46·000 2707	50·002 6782	31
32	37·494 0678	40·688 2880	44·227 0296	48·150 2775	52·502 7585	32
33	38·869 0085	42·298 6123	46·111 5702	50·354 0344	55·077 8413	33
34	40·257 6986	43·933 0915	48·033 8016	52·612 8853	57·730 1765	34
35	41·660 2756	45·592 0879	49·994 4776	54·928 2074	60·462 0818	35
36	43·076 8784	47·275 9692	51·994 3672	57·301 4126	63·275 9443	36
37	44·507 6471	48·985 1087	54·034 2545	59·733 9479	66·174 2226	37
38	45·952 7246	50·719 8854	56·114 9356	62·227 2966	69·159 4493	38
39	47·412 2508	52·480 6837	58·237 2384	64·782 9791	72·234 2328	39
40	48·886 3734	54·267 8939	60·401 9832	67·402 5535	75·401 2597	40
41	50·375 2371	56·081 9123	62·610 0228	70·087 6194	78·663 2975	41
42	51·878 9895	57·923 1410	64·862 2233	72·839 8078	82·023 1965	42
43	53·397 7704	59·791 9881	67·159 4678	75·660 8030	85·483 8923	43
44	54·931 7572	61·688 8679	69·502 6571	78·552 3231	89·048 4091	44
45	56·481 0747	63·614 2010	71·892 7103	81·516 1312	92·719 8614	45
46	58·045 8855	65·568 4140	74·330 5645	84·554 0344	96·501 4572	46
47	59·626 3443	67·551 9402	76·817 1758	87·667 8853	100·396 5010	47
48	61·222 6078	69·565 2193	79·353 5193	90·859 5824	104·408 3930	48
49	62·834 8338	71·608 6976	81·940 5897	94·131 0720	108·540 6478	49
50	64·463 1822	73·682 8280	84·579 4014	97·484 3488	112·796 8673	50
60	81·669 6699	96·214 6517	114·051 5394	135·991 5900	163·053 4368	60
70	100·676 3368	122·363 7530	149·977 9111	185·284 1142	230·594 0637	70
80	121·671 5217	152·710 8525	193·771 9578	248·382 7126	321·363 0186	80
90	144·863 2675	187·929 9004	247·156 6563	329·154 2533	443·348 9036	90
100	170·481 3829	228·803 0433	312·232 3059	432·548 6540	607·287 7327	100

TABLE VII. Amount of Annuity: $s_{\overline{n}|} = \dfrac{(1+i)^{-n}-1}{i}$. *(Continued)*

n	3½ per cent	4 per cent	4½ per cent	5 per cent	6 per cent	n
1	1·000 0000	1·000 0000	1·000 0000	1·000 0000	1·000 0000	1
2	2·035 0000	2·040 0000	2·045 0000	2·050 0000	2·060 0000	2
3	3·106 2250	3·121 6000	3·137 0250	3·152 5000	3·183 6000	3
4	4·214 9429	4·246 4640	4·278 1911	4·310 1250	4·374 6160	4
5	5·362 4659	5·416 3226	5·470 7097	5·525 6313	5·637 0930	5
6	6·550 1522	6·632 9755	6·716 8916	6·801 9128	6·975 3186	6
7	7·779 4075	7·898 2945	8·019 1518	8·142 0085	8·393 8376	7
8	9·051 6867	9·214 2263	9·380 0136	9·549 1089	9·897 4679	8
9	10·368 4958	10·582 7954	10·802 1142	11·026 5643	11·491 3160	9
10	11·731 3932	12·006 1071	12·288 2094	12·577 8925	13·180 7949	10
11	13·141 9919	13·486 3514	13·841 1788	14·206 7872	14·971 6426	11
12	14·601 9616	15·025 8055	15·464 0318	15·917 1265	16·869 9412	12
13	16·113 0303	16·626 8377	17·159 9133	17·712 9828	18·882 1377	13
14	17·676 9864	18·291 9112	18·932 1094	19·598 6320	21·015 0659	14
15	19·295 6809	20·023 5886	20·784 0543	21·578 5636	23·275 9699	15
16	20·971 0297	21·824 5311	22·719 3367	23·657 4918	25·672 5281	16
17	22·705 0153	23·697 5124	24·741 7069	25·840 3664	28·212 8798	17
18	24·499 6913	25·645 4129	26·855 0837	28·132 3857	30·905 6526	18
19	26·357 1805	27·671 2294	29·063 5625	30·539 0039	33·759 9917	19
20	28·279 6818	29·778 0786	31·371 4228	33·065 9541	36·785 5912	20
21	30·269 4707	31·969 2017	33·783 1368	35·719 2518	39·992 7257	21
22	32·328 9022	34·247 9698	36·303 3780	38·505 2144	43·392 2903	22
23	34·460 4137	36·617 8886	38·937 0300	41·430 4751	46·995 8277	23
24	36·666 5282	39·082 6041	41·689 1963	44·501 9989	50·815 5774	24
25	38·949 8567	41·645 9083	44·565 2102	47·727 0988	54·864 5120	25
26	41·313 1017	44·311 7446	47·570 6436	51·113 4538	59·156 3827	26
27	43·759 0602	47·084 2144	50·711 3236	54·669 1264	63·705 7657	27
28	46·290 6273	49·967 5870	53·993 3332	58·402 5828	68·528 1116	28
29	48·910 7993	52·966 2863	57·423 0332	62·322 7119	73·639 7983	29
30	51·622 6773	56·084 9378	61·007 0697	66·438 8495	79·058 1862	30
31	54·429 4710	59·328 3353	64·752 3878	70·760 7899	84·801 6774	31
32	57·334 5025	62·701 4687	68·666 2452	75·298 8294	90·889 7780	32
33	60·341 2100	66·209 5274	72·756 2263	80·063 7708	97·343 1647	33
34	63·453 1524	69·857 9085	77·030 2565	85·066 9594	104·183 7516	34
35	66·674 0127	73·652 2249	81·496 6180	90·320 3074	111·434 7799	35
36	70·007 6032	77·598 3138	86·163 9658	95·836 3227	119·120 8667	36
37	73·457 8693	81·702 2464	91·041 3443	101·628 1389	127·268 1187	37
38	77·028 8947	85.970 3363	96·138 2048	107·709 5458	135·904 2058	38
39	80·724 9060	90·409 1497	101·464 4240	114·095 0231	146·058 4581	39
40	84·550 2778	95·025 5157	107·030 3231	120·799 7742	154·761 9656	40
41	88·509 5375	99·826 5363	112·846 6876	127·839 7630	165·047 6836	41
42	92·607 3713	104·819 5978	118·924 7885	135·231 7511	175·950 5446	42
43	96·848 6293	110·012 3817	125·276 4040	142·993 3387	187·507 5772	43
44	101·238 3313	115·412 8770	131·913 8422	151·143 0056	199·758 0319	44
45	105·781 6729	121·029 3920	138·849 9651	159·700 1559	212·743 5138	45
46	110·484 0314	126·870 5677	146·098 2135	168·685 1637	226·508 1246	46
47	115·350 9726	132·945 3904	153·672 6331	178·119 4218	241·098 6121	47
48	120·388 2566	139·263 2060	161·587 9016	188·025 3929	256·564 5288	48
49	125·601 8456	145·833 7343	169·859 3572	198·426 6696	272·958 4006	49
50	130·997 9102	152·667 0837	178·503 0283	209·347 9957	290·335 9046	50
60	196·516 8829	237·990 6852	289·497 9540	353·583 7179	533·128 1809	60
70	288·937 8646	364·290 4588	461·869 6796	588·528 5107	967·932 1696	70
80	419·306 7868	551·244 9768	729·557 6985	971·228 8213	1746·599 8914	80
90	603·205 0270	827·983 3335	1145·269 0066	1594·607 3010	3141·075 1872	90
100	862·611 6567	1237·623 7046	1790·855 9563	2610·025 1569	5638·368 0586	100

TABLE VIII. Present Value of an Annuity: $a_{\overline{n}|} = \dfrac{1 - (1 + i)^{-n}}{i}$

n	1 per cent	1½ per cent	2 per cent	2½ per cent	3 per cent	n
1	0·990 0990	0·985 2219	0·980 3922	0·975 6097	0·970 8738	1
2	1·970 3951	1·955 8834	1·941 5609	1·927 4242	1·913 4697	2
3	2·940 9853	2·912 2004	2·883 8833	2·856 0236	2·828 6113	3
4	3·901 9656	3·854 3846	3·807 7287	3·761 9742	3·717 0984	4
5	4·853 4312	4·782 6450	4·713 4595	4·645 8285	4·579 7072	5
6	5·795 4765	5·697 1872	5·601 4309	5·508 1254	5·417 1914	6
7	6·728 1945	6·598 2140	6·471 9911	6·349 3906	6·230 2830	7
8	7·651 6778	7·485 9251	7·325 4811	7·170 1372	7·019 6922	8
9	8·566 0176	8·360 5173	8·162 2367	7·970 8655	7·786 1089	9
10	9·471 3045	9·222 1846	8·982 5850	8·752 0639	8·530 2028	10
11	10·367 6282	10·071 1178	9·786 8480	9·514 2087	9·252 6242	11
12	11·255 0775	10·907 5052	10·575 3412	10·257 7646	9·954 0040	12
13	12·133 7401	11·731 5322	11·348 3738	10·983 1850	10·634 9553	13
14	13·003 7030	12·543 3815	12·106 2488	11·690 9122	11·296 0731	14
15	13·865 0525	13·343 2330	12·849 2635	12·381 3777	11·937 9351	15
16	14·717 8738	14·131 2640	13·577 7093	13·055 0027	12·561 1020	16
17	15·562 2513	14·907 6493	14·291 8719	13·712 1977	13·166 1185	17
18	16·398 2686	15·672 5609	14·992 0312	14·353 3636	13·753 5131	18
19	17·226 0085	16·426 1684	15·678 4620	14·978 8913	14·323 7991	19
20	18·045 5530	17·168 6388	16·351 4333	15·589 1623	14·877 4749	20
21	18·856 9831	17·900 1367	17·011 2092	16·184 5484	15·415 0241	21
22	19·660 3793	18·620 8247	17·658 0482	16·765 4132	15·936 9166	22
23	20·455 8211	19·330 8614	18·292 2041	17·332 1105	16·443 6084	23
24	21·243 3870	20·030 4054	18·913 9254	17·884 9858	16·935 5421	24
25	22·023 1557	20·719 6112	19·523 4565	18·424 3764	17·413 1477	25
26	22·795 2037	21·398 6317	20·121 0358	18·950 6111	17·876 8424	26
27	23·559 6076	22·067 6175	20·706 8978	19·464 0109	18·327 0315	27
28	24·316 4432	22·726 7167	21·281 2724	19·964 8887	18·764 1082	28
29	25·065 7853	23·376 0756	21·844 3847	20·453 5499	19·188 4546	29
30	25·807 7082	24·015 8380	22·396 4556	20·930 2926	19·600 4414	30
31	26·542 2854	24·646 1458	22·937 7015	21·395 4074	20·000 4285	31
32	27·269 5895	25·267 1387	23·468 3348	21·849 1780	20·388 7655	32
33	27·989 6926	25·878 9544	23·988 5636	22·291 8809	20·765 7918	33
34	28·702 6659	26·481 7285	24·498 5917	22·723 7863	21·131 8367	34
35	29·408 5801	27·075 5946	24·998 6193	23·145 1573	21·487 2204	35
36	30·107 5050	27·660 6843	25·488 8425	23·556 2511	21·832 2525	36
37	30·799 5099	28·237 1274	25·969 4534	23·957 3181	22·167 2354	37
38	31·484 6633	28·805 0516	26·440 6406	24·348 6030	22·492 4616	38
39	32·163 0330	29·364 5829	26·902 5888	24·730 3444	22·808 2151	39
40	32·834 6861	29·915 8452	27·355 4792	25·102 7750	23·114 7720	40
41	33·499 6892	30·458 9608	27·799 4894	25·466 1220	23·412 4000	41
42	34·158 1081	30·994 0500	28·234 7936	25·820 6068	23·701 3592	42
43	34·810 0081	31·521 2316	28·661 5623	26·166 4457	23·981 9021	43
44	35·455 4530	32·040 6222	29·079 9631	26·503 8495	24·254 2739	44
45	36·094 5084	32·552 3370	29·490 1599	26·833 0239	24·518 7125	45
46	36·727 2361	33·056 4898	29·892 3136	27·154 1696	24·775 4491	46
47	37·353 6991	33·553 1920	30·286 5820	27·467 4825	25·024 7078	47
48	37·973 9595	34·042 5536	30·673 1196	27·773 1537	25·266 7066	48
49	38·588 0787	34·524 6834	31·052 0780	28·071 3695	25·501 6569	49
50	39·196 1175	34·999 6881	31·423 6054	28·362 3117	25·729 7640	50
60	44·955 0384	39·380 2689	34·760 8867	30·908 6565	27·675 5637	60
70	50·168 5144	43·154 8718	37·498 6193	32·897 8570	29·123 4214	70
80	54·888 2061	46·407 3235	39·744 5136	34·451 8172	30·200 7634	80
90	59·160 8815	49·209 8345	41·586 9292	35·665 7685	31·002 4071	90
100	63·028 8788	51·624 7037	43·098 3516	36·614 1053	31·598 9053	100

TABLE VIII. Present Value of an Annuity: $a_{\overline{n}|} = \dfrac{1 - (1+i)^{-n}}{i}$. *(Continued)*

n	3½ per cent	4 per cent	4½ per cent	5 per cent	6 per cent	n
1	0·966 1836	0·961 5384	0·956 9378	0·952 3810	0·943 3960	1
2	1·899 6943	1·886 0947	1·872 6678	1·859 4103	1·833 3297	2
3	2·801 6370	2·775 0910	2·748 9644	2·723 2480	2·673 0120	3
4	3·673 0792	3·629 8952	3·587 5257	3·545 9505	3·465 1056	4
5	4·515 0525	4·451 8223	4·389 9767	4·329 4767	4·212 3638	5
6	5·328 5530	5·242 1369	5·157 8724	5·075 6921	4·917 3243	6
7	6·114 5440	6·002 0547	5·892 7009	5·786 3734	5·582 3814	7
8	6·873 9555	6·732 7449	6·595 8861	6·463 2128	6·209 7938	8
9	7·607 6865	7·435 3316	7·268 7905	7·107 8217	6·801 6923	9
10	8·316 6053	8·110 8958	7·912 7182	7·721 7349	7·360 0871	10
11	9·001 5510	8·760 4767	8·528 9169	8·306 4142	7·886 8746	11
12	9·663 3343	9·385 0738	9·118 5808	8·863 2516	8·383 8439	12
13	10·302 7385	9·985 6478	9·682 8524	9·393 5730	8·852 6830	13
14	10·920 5203	10·563 1229	10·222 8253	9·898 6409	9·294 9839	14
15	11·517 4109	11·118 3874	10·739 5457	10·379 6580	9·712 2490	15
16	12·094 1168	11·652 2956	11·234 0150	10·837 7696	10·105 8953	16
17	12·651 3206	12·165 6684	11·707 1914	11·274 0662	10·477 2597	17
18	13·189 6817	12·659 2970	12·159 9918	11·689 5869	10·827 6035	18
19	13·709 8374	13·133 9394	12·593 2936	12·085 3209	11·158 1165	19
20	14·212 4033	13·590 3263	13·007 9364	12·462 2103	11·469 9212	20
21	14·697 9742	14·029 1600	13·404 7239	12·821 1527	11·764 0766	21
22	15·167 1248	14·451 1153	13·784 4248	13·163 0022	12·041 5815	22
23	15·620 4108	14·856 8417	14·147 7749	13·488 5739	12·303 3790	23
24	16·058 3676	15·246 9631	14·495 4784	13·798 6418	12·550 3575	24
25	16·481 5146	15·622 0799	14·828 2090	14·093 9446	12·783 3562	25
26	16·890 3523	15·982 7692	15·146 6113	14·375 1853	13·003 1662	26
27	17·285 3645	16·329 5858	15·451 3028	14·643 0336	13·210 5341	27
28	17·667 0188	16·663 0632	15·742 8735	14·898 1273	13·406 1643	28
29	18·035 7670	16·983 7146	16·021 8885	15·141 0736	13·590 7210	29
30	18·392 0454	17·292 0333	16·288 8885	15·372 4510	13·764 8312	30
31	18·736 2758	17·588 4936	16·544 3910	15·592 8105	13·929 0860	31
32	19·068 8655	17·873 5515	16·788 8909	15·802 6767	14·084 0434	32
33	19·390 2082	18·147 6457	17·022 8621	16·002 5492	14·230 2296	33
34	19·700 6842	18·411 1978	17·246 7580	16·192 9040	14·368 1411	34
35	20·000 6619	18·664 6132	17·461 0124	16·374 1943	14·498 2459	35
36	20·290 4938	18·908 2820	17·666 0406	16·546 8517	14·620 9871	36
37	20·570 5254	19·142 5788	17·862 2398	16·711 2873	14·736 7803	37
38	20·841 0874	19·367 8642	18·049 9902	16·867 8927	14·846 0192	38
39	21·102 4999	19·584 4848	18·229 6557	17·017 0407	14·949 0747	39
40	21·355 0723	19·792 7735	18·401 5844	17·159 0864	15·046 2969	40
41	21·599 1037	19·993 0518	18·566 1095	17·294 3680	15·138 0159	41
42	21·834 8828	20·185 6267	18·723 5498	17·423 2076	15·224 5433	42
43	22·062 6887	20·370 7949	18·874 2103	17·545 9120	15·306 1729	43
44	22·282 7910	20·548 8413	19·018 3830	17·662 7733	15·383 1820	44
45	22·495 4503	20·720 0397	19·156 3474	17·774 0691	15·455 8321	45
46	22·700 9181	20·884 6536	19·288 3707	17·880 0665	15·524 3699	46
47	22·899 4378	21·042 9361	19·414 7088	17·981 0157	15·589 0282	47
48	23·091 2443	21·195 1309	19·535 6065	18·077 1578	15·650 0261	48
49	23·276 5645	21·341 4720	19·651 2981	18·168 7217	15·707 5723	49
50	23·455 6179	21·482 1846	19·762 0078	18·255 9255	15·761 8606	50
60	24·944 7341	22·623 4900	20·638 0220	19·829 2895	16·161 4277	60
70	26·000 3966	23·394 5150	21·202 1119	19·342 6766	16·384 5439	70
80	26·748 7757	23·915 3918	21·565 3449	19·596 4605	16·509 1308	80
90	27·279 3156	24·267 2776	21·799 2408	19·752 2617	16·578 6994	90
100	27·655 4254	24·504 9990	21·949 8527	19·847 9102	16·617 5462	100

TABLE IX. Periodic Payments of an Annuity: $\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$

n	1 per cent	1½ per cent	2 per cent	2½ per cent	3 per cent	n
1	1·010 0000	1·015 0000	1·020 0000	1·025 0000	1·030 0000	1
2	0·507 5124	0·511 2779	0·515 0495	0·518 8272	0·522 6108	2
3	0·340 0221	0·343 3830	0·346 7547	0·350 1372	0·353 5304	3
4	0·256 2811	0·259 4448	0·262 6238	0·265 8179	0·269 0270	4
5	0·206 0398	0·209 0893	0·212 1584	0·215 2469	0·218 3546	5
6	0·172 5484	0·175 5252	0·178 5258	0·181 5500	0·184 5975	6
7	0·148 6283	0·151 5562	0·154 5120	0·157 4954	0·160 5064	7
8	0·130 6903	0·133 5840	0·136 5098	0·139 4674	0·142 4564	8
9	0·116 7404	0·119 6098	0·122 5154	0·125 4569	0·128 4339	9
10	0·105 5821	0·108 4342	0·111 3265	0·114 2588	0·117 2305	10
11	0·096 4541	0·099 2938	0·102 1779	0·105 1060	0·108 0774	11
12	0·088 8488	0·091 6800	0·094 5596	0·097 4871	0·100 4621	12
13	0·082 4148	0·085 2404	0·088 1184	0·091 0483	0·094 0295	13
14	0·076 9012	0·079 7233	0·082 6020	0·085 5365	0·088 5263	14
15	0·072 1238	0·074 9444	0·077 8255	0·080 7665	0·083 7666	15
16	0·067 9446	0·070 7651	0·073 6501	0·076 5990	0·079 6108	16
17	0·064 2581	0·067 0796	0·069 9698	0·072 9278	0·075 9525	17
18	0·060 9820	0·063 8058	0·066 7021	0·069 6701	0·072 7087	18
19	0·058 0518	0·060 8785	0·063 7818	0·066 7606	0·069 8139	19
20	0·055 4153	0·058 2457	0·061 1567	0·064 1471	0·067 2157	20
21	0·053 0308	0·055 8655	0·058 7848	0·061 7873	0·064 8718	21
22	0·050 8637	0·053 7033	0·056 6314	0·059 6466	0·062 7474	22
23	0·048 8858	0·051 7308	0·054 6681	0·057 6964	0·060 8139	23
24	0·047 0735	0·047 9241	0·052 8711	0·055 9128	0·059 0474	24
25	0·045 4068	0·048 2634	0·051 2204	0·054 2759	0·057 4279	25
26	0·043 8689	0·046 7320	0·049 6992	0·052 7688	0·055 9383	26
27	0·042 4455	0·045 3153	0·048 2931	0·051 3769	0·054 5642	27
28	0·041 1244	0·044 0011	0·046 9897	0·050 0879	0·053 2932	28
29	0·039 8950	0·042 7788	0·045 7784	0·048 8913	0·052 1147	29
30	0·038 7481	0·041 6392	0·044 6499	0·047 7776	0·051 0193	30
31	0·037 6757	0·040 5743	0·043 5964	0·046 7390	0·049 9989	31
32	0·036 6709	0·039 5771	0·042 6106	0·045 7683	0·049 0466	32
33	0·035 7274	0·038 6414	0·041 6865	0·044 8594	0·048 1561	33
34	0·034 8400	0·037 7619	0·040 8187	0·044 0068	0·047 3220	34
35	0·034 0037	0·036 9336	0·040 0022	0·043 2056	0·046 5393	35
36	0·033 2143	0·036 1524	0·039 2328	0·042 4516	0·045 8038	36
37	0·032 4680	0·035 4144	0·038 5068	0·041 7409	0·045 1116	37
38	0·031 7615	0·034 7161	0·037 8206	0·041 0701	0·044 4593	38
39	0·031 0916	0·034 0546	0·037 1711	0·040 4362	0·043 8438	39
40	0·030 4556	0·033 4271	0·036 5558	0·039 8362	0·043 2624	40
41	0·029 8510	0·032 8311	0·035 9719	0·039 2679	0·042 7124	41
42	0·029 2756	0·032 2643	0·035 4173	0·038 7288	0·042 1917	42
43	0·028 7274	0·031 7246	0·034 8899	0·038 2169	0·041 6981	43
44	0·028 2044	0·031 2104	0·034 3879	0·037 7304	0·041 2298	44
45	0·027 7050	0·030 7198	0·033 9096	0·037 2675	0·040 7852	45
46	0·027 2278	0·030 2512	0·033 4534	0·036 8268	0·040 3625	46
47	0·026 7711	0·029 8034	0·033 0179	0·036 4067	0·039 9605	47
48	0·026 3338	0·029 3750	0·032 6018	0·036 0060	0·039 5778	48
49	0·025 9147	0·028 9648	0·032 2040	0·035 6235	0·039 2131	49
50	0·025 5127	0·028 5717	0·031 8232	0·035 2581	0·038 8655	50
60	0·022 2444	0·025 3934	0·028 7680	0·032 3534	0·036 1330	60
70	0·019 9328	0·023 1724	0·026 6676	0·030 3971	0·034 3366	70
80	0·018 2188	0·021 5483	0·025 1607	0·029 0260	0·033 1118	80
90	0·016 9031	0·020 3211	0·024 0460	0·028 0381	0·032 2556	90
100	0·015 8657	0·019 3706	0·023 2027	0·027 3119	0·031 6467	100

TABLE IX. Periodic Payments of an Annuity: $\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i$. *(Continued)*

n	3½ per cent	4 per cent	4½ per cent	5 per cent	6 per cent	n
1	1·035 0000	1·040 0000	1·045 0000	1·050 0000	1·060 0000	1
2	0·526 4005	0·530 1961	0·533 9976	0·537 8049	0·545 4369	2
3	0·356 9342	0·360 3485	0·363 7734	0·367 2086	0·374 1098	3
4	0·272 2511	0·275 4900	0·278 7436	0·282 0118	0·288 5915	4
5	0·221 4814	0·224 6271	0·227 7916	0·230 9748	0·237 3964	5
6	0·187 6682	0·190 7619	0·193 8784	0·197 0175	0·203 3626	6
7	0·163 5445	0·166 6096	0·169 7015	0·172 8198	0·179 1350	7
8	0·145 4766	0·148 5278	0·151 6096	0·154 7218	0·161 0359	8
9	0·131 4460	0·134 4930	0·137 5745	0·140 6901	0·147 0222	9
10	0·120 2414	0·123 2909	0·126 3788	0·129 5046	0·135 8680	10
11	0·111 0920	0·114 1490	0·117 2482	0·120 3889	0·126 7929	11
12	0·103 4840	0·106 5522	0·109 6662	0·112 8254	0·119 2770	12
13	0·097 0616	0·100 1437	0·103 2754	0·106 4558	0·112 9601	13
14	0·091 5707	0·094 6690	0·097 8203	0·101 0240	0·107 5849	14
15	0·086 8251	0·089 9411	0·093 1138	0·096 3423	0·102 9628	15
16	0·082 6848	0·085 8200	0·089 0154	0·092 2699	0·098 9521	16
17	0·079 0431	0·082 1985	0·085 4176	0·088 6991	0·095 4448	17
18	0·075 8168	0·078 9933	0·082 2369	0·085 5462	0·092 3565	18
19	0·072 9403	0·076 1386	0·079 4073	0·082 7450	0·089 6209	19
20	0·070 3611	0·073 5818	0·076 8761	0·080 2426	0·087 1846	20
21	0·068 0366	0·071 2801	0·074 6006	0·077 9961	0·085 0046	21
22	0·065 9321	0·069 1988	0·072 5456	0·075 9705	0·083 0456	22
23	0·064 0188	0·067 3091	0·070 6825	0·074 1368	0·081 2785	23
24	0·062 2728	0·065 5868	0·068 9870	0·072 4709	0·079 6790	24
25	0·060 6740	0·064 0120	0·067 4390	0·070 9525	0·078 2267	25
26	0·059 2054	0·062 5674	0·066 0214	0·069 5643	0·076 9044	26
27	0·057 8524	0·061 2385	0·064 7195	0·068 2919	0·075 6972	27
28	0·056 6026	0·060 0130	0·063 5208	0·067 1225	0·074 5926	28
29	0·055 4454	0·058 8799	0·062 4146	0·066 0455	0·073 5796	29
30	0·054 3713	0·057 8301	0·061 3915	0·065 0514	0·072 6489	30
31	0·053 3724	0·056 8554	0·060 4434	0·064 1321	0·071 7922	31
32	0·052 4415	0·055 9486	0·059 5632	0·063 2804	0·071 0023	32
33	0·051 5724	0·055 1036	0·058 7445	0·062 4900	0·070 2729	33
34	0·050 7597	0·054 3148	0·057 9819	0·061 7554	0·069 5984	34
35	0·049 9984	0·053 5773	0·057 2704	0·061 0717	0·068 9739	35
36	0·049 2842	0·052 8869	0·056 6058	0·060 4345	0·068 3948	36
37	0·048 6132	0·052 2396	0·055 9840	0·059 8398	0·067 8574	37
38	0·047 9821	0·051 6319	0·055 4017	0·059 2842	0·067 3581	38
39	0·047 3878	0·051 0608	0·054 8557	0·058 7646	0·066 8938	39
40	0·046 8273	0·050 5235	0·054 3432	0·058 2782	0·066 4615	40
41	0·046 2982	0·050 0174	0·053 8616	0·057 8223	0·066 0589	41
42	0·045 7983	0·049 5402	0·053 4087	0·057 3947	0·065 6834	42
43	0·045 3254	0·049 0899	0·052 9824	0·056 9933	0·065 3331	43
44	0·044 8777	0·048 6645	0·052 5807	0·056 6162	0·065 0061	44
45	0·044 4534	0·048 2625	0·052 2020	0·056 2617	0·064 7005	45
46	0·044 0511	0·047 8820	0·051 8447	0·055 9282	0·064 4148	46
47	0·043 6692	0·047 5219	0·051 5073	0·055 6142	0·064 1477	47
48	0·043 3065	0·047 1806	0·051 1886	0·055 3184	0·063 8977	48
49	0·042 9617	0·046 8571	0·050 8872	0·055 0396	0·063 6636	49
50	0·042 6337	0·046 5502	0·050 6022	0·054 7767	0·063 4443	50
60	0·040 0886	0·044 2018	0·048 4543	0·052 8282	0·126 7929	60
70	0·038 4610	0·042 7451	0·047 1651	0·051 6992	0·061 0331	70
80	0·037 3849	0·041 8141	0·046 3707	0·051 0296	0·060 5725	80
90	0·036 6578	0·041 2078	0·045 8732	0·050 6271	0·060 3184	90
100	0·036 1593	0·040 8080	0·045 5584	0·050 3831	0·060 1774	100

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Answers to Odd-Numbered Exercises

CHAPTER II

Article 19 (page 22)

3. (a) -15; (b) -5; (c) 36; (d) -180; (e) 9; (f) $\frac{4}{5}$; (g) $\frac{13}{20}$; (h) $-\frac{4}{3}$; (i) $-\frac{1}{2}$; (j) $-\frac{15}{4}$; (k) $-\frac{27}{20}$; (l) $\frac{14}{13}$. **5.** $4; -6$.

7. $(5 + a)x - (2 + 4a)y + (3a - 1)z$.

9. $(2y - 2x)(a - b)$; $(y^2 - x^2)(a - b)$; $(6)(-20)$.

11. $15b - 14a - 6$.

13. $8x - 40y + 34$.

15. $15c - 2b - 8a + 27$.

17. $3 - 12x$.

19. $18a - 33x + 9y - 6$.

21. $11x^2 - 7x - 3ax$; $4ax - 3a - 5a^2x$.

23. $6a^2 + 5b^2 + 7a - 6b - 12$.

25. $x + y$.

27. $7a - 13ab - a^2 - 5b$.

29. $15x - 2xy - 6y + 3$.

31. $x^2y + 5x^3y^2 - xy^2$.

33. $2ax - 2a^2x^2$.

35. $13c - 4b - 2a - 5ab - bc - 17$.

CHAPTER III

Article 22 (page 34)

1. (a) a^{10}; (b) a^4; (c) a^{-4}; (d) a^{21}; (e) a^{21}; (f) x^{3a+3}; (g) x^{4a}; (h) x^{3a-3}; (i) x^{2a}; (j) x^{9a}; (k) 2; (l) 4; (m) 8; (n) 16; (o) $\frac{1}{32}$; (p) $\frac{1}{2}$; (q) $\frac{1}{125}$; (r) 27; (s) $\frac{1}{9}$; (t) $\frac{1}{243}$; (u) 1; (v) 1; (w) 1; (x) 1; (y) 1.

3. (a) -4; (b) 12; (c) $\dfrac{1}{.027}$; (d) 16; (e) 3; (f) .00032.

5. (a) $\sqrt[3]{ab}$; (b) $\sqrt[3]{a^4}$; (c) $\sqrt[6]{243}$; (d) $\sqrt[6]{a^3b^2}$; (e) $\sqrt[3]{40}$; (f) $\sqrt[5]{-96}$; (g) $\sqrt[4]{15{,}552}$; (h) $\sqrt[6]{3}$; (i) $\sqrt[5]{6}$; (j) $\sqrt[5]{6}$; (k) $\sqrt[6]{a}$; (l) $\sqrt[12]{7}$.

7. 9.

9. $\frac{125}{64}$.

11. 9.

13. $-\sqrt{3}$.

15. $15\sqrt{5}$.

17. $\frac{21}{5}\sqrt[3]{5} - 5\sqrt{6}$.

19. $29\frac{5}{6}\sqrt{6}$.

21. $\frac{9}{10}\sqrt{10}$.

23. $-26\sqrt[3]{3}$.

25. $8\sqrt{5}$. **27.** $\frac{1}{x}\sqrt{2x}$. **29.** $3 - 2\sqrt{2}$.

31. $13 + 4\sqrt{3}$. **33.** 1. **35.** -11.

37. $-\sqrt{3} + \sqrt{2}$. **39.** $2x - 2\sqrt{x^2 - y^2}$. **41.** $\sqrt[3]{3}$; $\sqrt[3]{2}$; $\sqrt[5]{5}$.

43. $\sqrt{5}$; $\sqrt[3]{11}$; $\sqrt[6]{13}$. **47.** (a) 1,728; (b) 14,400; (c) 0.4489; (d) 0.300763; (e) 9,409; (f) 658,503; (g) 18; (h) 67.08; (i) 3.958; (j) 8.5270; (k) 18.371; (l) 0.2490.

Article 23 (page 43)

1. $\frac{x}{y}$. **3.** $\frac{y}{x}$. **5.** $x^2y^{-3/2}$.

7. $xy^{-1/2}$. **9.** $a^{7/4}b^{-17/12}$. **11.** $x^{-1/6}y^{1/4}$.

13. x^6y^4. **15.** $x^{b^2-a^2}$. **17.** $\frac{1}{4}$.

19. x^5. **21.** x^{a-1}. **23.** x.

25. $\frac{1}{9}$. **27.** a^{-5}. **29.** $x^{-2/5}y^{13/3}z^{-2/3}$.

31. $\sqrt[6]{a^5}$. **33.** $x^{7/8}$. **35.** $4a\sqrt[6]{2a^3b}$.

37. $6\sqrt[6]{48}$. **39.** $a^{-2/3}b^{-1/2}$. **41.** $x^{-1/2}y^{-1/3}z^{-1/6}$.

43. $\sqrt[6]{3}$. **45.** $x^2y^4z^{12}\sqrt{yz}$. **47.** $\frac{27}{8x^3y^3}$.

49. $x^{7/30}y^{3/10}$. **51.** $\frac{xz}{2abc^2}\sqrt[3]{18xy^2a^2bc}$. **53.** $2^{1/3}x^{3/2}b^{4/3}c^{1/18}$.

55. $\frac{ac}{b}\sqrt[6]{a^2b^2c^3}$. **57.** $231a^2b^3c^4\sqrt{c}$. **59.** $30\sqrt{10}$.

61. $320\sqrt{5}$. **63.** $\frac{y^3 - x^3}{x^3y^3}$. **65.** $\frac{y^3}{y^3 - x^3}$.

67. $\frac{x^3y^3}{y^3 - x^3}$. **69.** $\frac{1}{(x - y)^3}$. **71.** $\frac{x^3}{x^3 - 1}$.

73. $\frac{1}{2}\sqrt{2}$. **75.** $\frac{1}{10}\sqrt{30}$. **77.** $\frac{2}{3}\sqrt{2}$.

79. $\frac{1}{3}\sqrt[3]{18}$. **81.** $\frac{1}{2}\sqrt[3]{6}$. **83.** $\frac{1}{2}\sqrt[3]{3}$.

85. $\frac{1}{a}\sqrt[3]{a}$. **87.** $-2 - \sqrt{2}$. **89.** $\frac{19 + 8\sqrt{3}}{-13}$.

91. $\frac{35 + 12\sqrt{6}}{19}$. **93.** $\frac{2 + 5\sqrt{2} + 3\sqrt{3} - 4\sqrt{6}}{23}$.

Article 24 (page 47)

1. (a) $6x^5 - x^4 + 5x^3 - 4x + 2$; (b) $x^3 - 2x^{3/2} + 5x - x^{1/2} + 7$;
(c) $-2x^5 - 4x^4 + x^3 + 5 - 5x^{-2} + 3x^{-3}$;
(d) $(5 - a)x^2 + (b - 3)x + (2 - c)$;
(e) $(2 + a)x^3 - (5 + 3a)x^2 + (2 - a)x + (7 + b - b^3)$.

3. $6x^2 - 11x - 10$. **5.** $x^2 + 2xz + z^2 - y^2$.

7. $3x^4 - 2x^3y - 2x^2y^2 + 2xy^3 - y^4 + 6x^3 - 13x^2y + 8xy^2 - 3y^3$.
9. $\frac{1}{2}a^2 + \frac{19}{36}ab - \frac{5}{9}ac - \frac{1}{3}b^2 + \frac{11}{18}bc - \frac{2}{9}c^2$.
11. $x^{3/2} - 2x^{5/4} + 2x - 2x^{3/4} + x^{1/2}$.
13. $10a^6 - 11a^5b - 12a^4b^2 - 9a^3b^3 + 8a^2b^4 + 6ab^5 + 8b^6$.
15. $a^{3/2} - 5a^{5/2} + 11a^{3/2} - 16a^{3/2} + 12 - 9a^{-3/2}$.
17. $x^3 - x^2 - 3x - 1$.
19. $3x^2 - 2x - 5$.
21. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$.
23. $x^3 - 3x^2y + 5xy^2 - 10y^3$; remainder, $15xy^4 + 9y^5$.
25. $\frac{3}{4}x^3 + \frac{5}{4}x^2 + \frac{9}{8}x - \frac{47}{16}$; remainder, $-\frac{79}{16}x + \frac{47}{16}$.
27. $x^2 - x - y$.
29. $x^{1/2} + 1$.
31. $\frac{2}{3}$; remainder, $-\frac{5}{3}$.
33. $x^2y + (z - y)x + y$.
35. $\frac{3}{5}x^2 + \frac{1}{25}x + \frac{3}{125}$, remainder $-\frac{11}{125}x + \frac{17}{125}$.
37. $x^2y - (yz + z^2)x + yz^2$.
39. $x^{n-1} - x^{n-2}y + \cdots + xy^{n-2} - y^{n-1}$; remainder, $-2y^n$.
41. $x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}$; remainder, $2y^n$.

CHAPTER IV

Article 26 (page 56)

1. 4. 3. (a) 5; (b) 2; (c) 2; (d) 2. 5. $6x^2 - 13xy + 6y^2$.
7. $a^3 + 3a^2b^2 - 3ab - 9b^3$.
9. $ax^2 + x - a^2x - a$.
11. $x^2 + 4y^2 + 9z^2 - 4xy + 6xz - 12yz$.
13. $x^2 + 4y^2 + 9z^2 + 4xy - 6xz - 12yz$.
15. $15x^3 - 18x^2y^2 - 20xy + 24y^3$.
17. $e^{2x} + 2 + e^{-2x}$.
19. $x^3 - 27$.
21. $x^2 - y^2 + z^2 + 2xz$.
23. $6x - \sqrt{x} - 15$.
25. (a) $x(x + 1)(x - 1)$; (b) $(x - y)(x + y + a)$.
27. (a) $(x - 4)(x + 4)$; (b) $(x - 2)(x + 2)(x^2 + 4)$.
29. (a) $(x - 5y)^2$; (b) $(x^2 + 5y^2)^2$.
31. (a) $(x - 3)(x^2 + 3x + 9)$; (b) $(2x + 3y^2)(4x^2 - 6xy^2 + 9y^4)$.
33. (a) $(x + 1)^3$; (b) $(x - 2)^3$.
35. (a) $(a - 12)(a + 7)$; (b) $(1 - 2x)(1 + 3x)$.
37. (a) $(3x + 2y)(2x - 3y)$; (b) $(2x - 3)(x + 4)$.
39. (a) $(x - y - z)(x - y + z)$; (b) $(x - y + z)(x + y - z)$.
41. (a) $(x^2 + x + 1)(x^2 - x + 1)$;
(b) $(2x^2 + xy + y^2)(2x^2 - xy + y^2)$.
43. (a) $(x - y + 1)^2$; (b) $(x + y - 2)^2$.
45. (a) $(ax + 1)(x - a)$; (b) $(ax - 1)(x - a)$;
(c) $(x - a^2)(ax + 1)$.
47. $(4x^2 + 2xab^2 + a^2b^4)(4x^2 - 2xab^2 + a^2b^4)$.
49. $(3x - 2y^2)(2x + 3y^2)$.
51. $3x^2(4x - 9y^2)(4x + 9y^2)$.
53. $(xy - 10z)(xy + 9z)$.
55. $(3xy - 4z)(2xy + 5z)$.

57. $(x^2y + 5z^2)(x^2y + z^2)$.
59. $(a + b)(x - y)$.
61. $(2x - 3z - 2y)(2x - 3z + 2y)$.
63. $(x - 3a - y)(x - a)$.
65. $3(x^2 - 5ab^2)(x^4 + 5ax^2b^2 + 25a^2b^4)$.
67. $(a - b + c)[a^2 + a(b - c) + (b - c)^2]$.
69. $(2x + 5y^2)(2x - 3y^2)$.
71. $(x - 2y)^3$.
75. $(x^{1/2} - 3)(x^{1/2} + 2)$.
77. $(x^{5/2} - 2)(x^{5/2} + 4)$.
79. $\left(\frac{x}{y} + \frac{y}{x}\right)^2$.
81. $(x^{-1} + 3)^2$.
83. $(e^x + e^{-x})^2$.
85. $[(x + 2)^{1/2} - 1]^2$.
87. $c\left(\frac{1}{b} + \frac{1}{a}\right)^2$.
89. $\left(x + \frac{3}{x}\right)\left(x - \frac{2}{x}\right)$.
91. $(x^{1/2} - 1)(2x^{1/2} - 3)$.

Article 29 (page 64)

1. $\frac{2x + 9}{(x - 3)(x + 2)}$.
3. $\frac{1}{1 - a^2}$.
5. $\frac{(x - 3)(x + 5)}{x^2}$.
7. $\frac{a^2 + b^2}{(a + b)^2}$.
9. 252.
11. $\frac{x + 3y}{3x + y}$.
13. $a + b + x$.
15. $\frac{13}{3}$.
17. H.C.F., 2^2; L.C.M., $2^4 \cdot 3 \cdot 5^2$.
19. H.C.F., $2ax^2$; L.C.M., $2^2 \cdot 3^2a^3x^3y^2$.
21. H.C.F., 1; L.C.M., $(x - y)(x + y)^2$.
23. H.C.F., xy; L.C.M., $3 \cdot 5 \cdot 7x^2y^2(x - y)^2(x + y)^2$.
25. $\frac{16x - 25y}{24}$.
27. $\frac{2x^3 - 2x - 1}{x(x - 1)}$.
29. $\frac{10x^2 - 13x - 30}{6(1 - x)}$.
31. $\frac{x^5 - y^5}{x^6 - y^6}$.
33. $\frac{x^2 - 10x + 1}{(x - 2)(x - 3)(x + 1)}$.
35. $\frac{8y(x^2 - 2xy - x - 2y)}{(x + 2y)(x - 2y)^2}$.
37. $\frac{-6ax(x^2 + 2x + a^2)}{(x^2 - a^2)(x^2 + a^2)}$.
39. 0.
41. $\frac{8x^3 - 24x^2 + 12x + 27}{6x^2(2x - 3)}$.
43. $\frac{13x - 3x^2}{x^2 - 4}$.
45. $\frac{2a + 7b}{5a + 3b}$.
47. $\frac{a + b}{a^2 + b^2}$.

49. $(x - y)$.

51. $\frac{x+1}{x-1}$.

53. $a + b$.

55. $\frac{x^2y^2}{y^2 - x^2}$.

57. $-\frac{x+y}{y}$.

59. 1.

61. $\frac{(x-2)(x+2)^2}{4x(x+4)}$.

63. $\frac{x^2+1}{x+1}$.

65. $(x - 2y)^2$.

67. $\frac{x^2-1}{x^2+1}$.

69. $\frac{x+y}{y}$.

71. $\frac{(2x-7)(2x+1)}{2(4x^2-5x-7)}$.

73. $1 - x$.

75. $\frac{x^2-1}{x^3}$.

77. $\frac{x^2+1}{x^3}$.

79. $\frac{15}{7}$.

81. $\frac{a^2}{a^2 - x^2}$.

83. $\frac{a^2}{a^2 - x^2}$.

CHAPTER V

Article 30 (page 74)

1. 3. **3.** 2. **5.** -3. **7.** 0.
9. $\frac{3}{2}$. **11.** -2. **13.** 3. **15.** $\frac{1}{5}$.
17. 5. **19.** $\frac{1}{2}$. **21.** 81. **23.** $\sqrt[9]{3}$.
25. $\frac{1}{25}$. **27.** 2. **29.** 3. **31.** $\frac{1}{3}$.
33. 10,000. **35.** $\frac{5}{4}$. **37.** $\frac{4}{5}$. **39.** -3.
41. $\frac{8}{27}$. **43.** $\frac{27}{8}$. **45.** -2. **47.** -1.
49. 6. **51.** 4. **53.** 2. **55.** $\frac{1}{36}$.

57. 10. **59.** $\log 4 + \log \pi + 3 \log r - \log 3$.

61. $\log 5 + \frac{1}{2} \log 35 - 2 \log 12$.

63. $9[8 \log 7 + \frac{1}{7}(5 \log 19 + \frac{2}{3} \log 13 + \frac{2}{3} \log \pi) - (3 \log 23 + 4 \log 17)]$.

65. $2 \log 5 + 2 \log \pi$. **67.** $\log 35 - \log 22$.

69. $6 \log 37 + \frac{1}{3}(4 \log 23 + 5 \log 47 - 8 \log 17)$.

71. $\log 2 + \log \pi + \frac{1}{2} \log l - \frac{1}{2} \log g$.

73. $\log 15 + \frac{1}{3} \log 71 - \log 201 - 2 \log 67$.

75. $3 \log 11 + \frac{1}{2} \log 78 - \log 7 - 6 \log 5 - \frac{1}{4} \log 22$.

77. $\log 8 + \log 11 + \log 107 + \log 205 - \log 65 - \log 72$.

79. $\frac{1}{3}(2 \log 24 + 5 \log 4 - 5 \log 7) - 5(3 \log 21 + \frac{1}{2} \log 71 - \frac{1}{2} \log 22)$.

81. $4 \log 27 + \frac{1}{7}(2 \log 28 - 3 \log 101) - 5 \log 61 + 4 \log 84$.

83. $\frac{1}{9} \log 367 - \frac{1}{9} \log 231 + 7 \log 67 - 7 \log 23 - \frac{5}{3} \log 123 + \frac{3}{8} \log 431$.

85. $x = \frac{2 \cdot 7}{5^3}$. **87.** $x = \frac{a^2 \cdot d^5}{\sqrt[4]{b^3} \cdot \sqrt[3]{c}}$.

89. $\log A = \log \pi + 2 \log r$. **91.** $\log V = \log 2 + \log p - \log d$.

93. (a) $\log Q = \log \pi + \log p + 4 \log r + \log T - \log 8 - \log l - \log n$; (b) $\log T = \log Q - \log \pi - \log p - 4 \log r + \log 8 + \log l + \log n$.

Article 33 (page 83)

1. (a) 3,000; (b) 26,500; (c) 0.0000672; (d) 0.0265; (e) 30,000; (f) 0.0000000003; (g) 0.0376; (h) 0.0000035; (i) 60; (j) 0.00005893; (k) 29,990,000,000; (l) 0.0000455; (m) 0.000000058.

3. (a) 4; (b) -2; (c) 3; (d) -4; (e) 7; (f) -1; (g) 0; (h) -2; (i) -7; (j) 1; (k) 0; (l) -1.

5. (a) 3.5729; (b) 0.9253; (c) $0.8048 - 1$; (d) $0.3747 - 2$; (e) 0.0212; (f) 3.9031; (g) 2.4771; (h) 2.0253; (i) $0.7782 - 2$.

7. (a) 0.4278; (b) 0.5669; (c) 1.4273; (d) 3.9530; (e) $0.8763 - 1$; (f) $0.8129 - 3$; (g) 3.3017; (h) $0.8375 - 2$; (i) 1.4586.

9. (a) 4.09; (b) 70; (c) 794.5; (d) 2,899; (e) 1.806; (f) 0.2819; (g) 4,836; (h) 0.001006; (i) 1,834,000.

11. (a) $9.5722 - 10$; (b) $9.4331 - 10$; (c) $8.5727 - 10$; (d) $6.0470 - 10$; (e) 0.1237; (f) 2.1871; (g) $6.6983 - 10$; (h) 1.1625; (i) $8.5414 - 10$.

Article 34 (page 89)

1. 49,270. **3.** 0.3577. **5.** 24. **7.** 0.3557.
9. 1.536. **11.** 0.04054. **13.** 1.707. **15.** 0.06214.
17. 0.1702. **19.** 169. **21.** 5.805. **23.** 1.790.
25. 1,635,000.

The following answers were obtained with a five-place table of logarithms.

27. 2.3501. **29.** 144.38. **31.** 1.2762. **33.** 9,349.2.
35. 1,861.2. **37.** 158.13. **39.** 0.40390. **41.** 12.182.
43. 0.81873. **45.** 0.2591. **47.** 0.58817. **49.** 1.5950.
51. 4.4394. **53.** 122,880,000 square feet.
55. \$1,883.00, \$1,113.60. **57.** \$14,889, \$18,392.80.
59. 36,638,000 square feet. **61.** \$13,603.
63. 0.19744 centimeter. **65.** 99.295 centimeters.
67. (a) \$1,806.10, (b) \$1,814.00, (c) \$1,819.40.
69. 0.48828. **71.** (a) 6,208 H.P., (b) 2.912.7 H.P.
73. 0.04809 henry. **75.** (a) 260.9° F., (b) 7.24 pounds.
77. 11,093.3. **79.** 0.40307. **81.** 55.906.

Article 35 (page 95)

1. 2.322. **3.** 0.4307. **5.** 2.57. **7.** 0.9031.
9. 2.079. **11.** -1.966. **13.** 0.3562. **15.** 1.131.

17. -3. **19.** $\frac{1}{7}$. **21.** -2. **23.** 2.617.
25. 3.101. **27.** 5.22 **29.** 354.

Article 36 (page 97)

1. (a) 0.69315; (b) 1.09861; (c) 1.60944; (d) 1.94591.
3. (a) 1.38629; (b) 1.79176; (c) 2.07944; (d) 2.19722; (e) 2.30259.
5. (a) 4.60518; (b) -2.30259; (c) -4.60518.

CHAPTER VI

Article 37 (page 102)

1. (c) 122.8×10^6 in 1930; 131.7×10^6 in 1940.
3. Rural, 22.4, 23.2; cities, 31.2, 31.1.
5. (a) 94,089, 91,914, 84,721, 62,104, 51,230; (b) 16.8, 47.6, 61.8, 73.5, 82.2, 92.8.
7. 1.22; 1.48; 1.80; 3.92; 7.76; 14.2 years; 22.5 years.

Article 38 (page 108)

1. -3; 15; 1; -5; $a^2 - 5a + 1$.
3. $\frac{1}{2}$; 1; $-\frac{14}{5}$; $3y - 1$; 0.
5. 4; 0; $-\frac{1}{6}$; ∞; -1.
7. 3; -17; $-\frac{3}{4}$; $\frac{23}{9}$.
9. $4x^2 - 1$; $x^4 - 1$; $\frac{1}{x^2} - 1$.
11. $\frac{x-2}{x^2+2x+3}$; $\frac{x-5}{x^2-4x+6}$; $\frac{2x-12}{x^2+8}$.
13. $\frac{1}{4}$; $\frac{2x^2-8x+5}{4(x-2)}$; $\frac{2x^4-3}{4x^2}$.
15. $x + \frac{1}{x}$; $x^2 + \frac{1}{x^2}$; $x^2 + 2 + \frac{1}{x^2}$.
17. $x + 2$; $x^2 + 2x + 2$; $\frac{x+2}{x+1}$.
19. $A = f(b, h) = bh$.
21. $A = f(r) = \pi r^2$.
23. $V = f(r) = \frac{4}{3}\pi r^3$.
25. $V = f(r,h) = \frac{1}{3}\pi r^2 h$.
27. (a) $F = \frac{9}{5}C + 32°$; (b) $C = \frac{5}{9}(F - 32°)$; (c) 25° C.

Article 39 (page 111)

3. (2, 7); (2, -1); (-6, 7); (-6, -1).
5. (-3, 2), (2, 2), (-3, -7); (-3, 11), (2, 11), (2, 2); (6, 2), (6, -3), (-3, -3); (-3, 7), (6, 7), (6, 2). **7.** y-axis; $x = 0$.
9. A line parallel to y-axis and 5 units to the right; a line parallel to y-axis and 7 units to the left; $x = 5$; $x = -7$.
11. A line bisecting quadrants I and III; $x = y$. **13.** $y = 2x$.

Article 41 (page 128)

1. 3. **3.** -3. **5.** $-4x^{-5} - 8x^{-3} + 7$.
7. $2x$. **9.** $2x - 5$. **11.** $-15x^{-6} - 6x^{-4} + 8x^{-3}$.

13. $2x - 2$. **15.** $2x + 2$. **17.** $-8x^{-5} - 9x^2 + x^{-\frac{1}{2}}$.
19. 0. **21.** 6. **23.** -4.
25. 16. **27.** (3, -16) minimum point. **29.** (-1, 9) maximum point.
31. (-1, 1) maximum point; (1, -7) minimum point. **33.** (2, 10) maximum point; (0, 6) minimum point.
35. (0, 8) maximum point; (2, -4) minimum point.
39. 10; 10. **41.** 1,250 square feet.
43. (a) $V = V_0 - gt$; (b) $\frac{V_0^2}{2g}$ feet.

CHAPTER VII

Article 46 (page 142)

3. (a) 5; (b) $-\frac{2}{3}$; (c) 2; (d) $\frac{4}{3}$.
5. Yes.
7. -4.
9. $\frac{9}{4}$.
11. $-\frac{17}{12}$.
13. $\frac{11}{6}$.
15. $\frac{2}{7}$.
17. No solution.
19. No solution.
21. $\frac{7}{8}$.
23. -10.
25. $\frac{2}{3}$.
27. $-\frac{1}{2}$.
29. No solution.
31. $-\frac{1}{3}a$.

33. $\frac{c}{a+b}$.

35. $\frac{a^2 - ab + b^2}{b - a}$.

37. $\frac{a^2 + b^2}{2a}$.

39. $\frac{ab(a + b + 2)}{a + 2ab - b}$.

41. $\frac{2ab}{a+b}$.

43. $\frac{ab - d}{ab + c}$.

45. $a = l - (n-1)d$; $n = \frac{l - a + d}{d}$; $d = \frac{l - a}{n - 1}$.

47. $P = \frac{A}{1 + in}$; $i = \frac{A - P}{Pn}$; $n = \frac{A - P}{Pi}$.

49. $a = rl - rs + s$; $r = \frac{s - a}{s - l}$; $l = \frac{rs - s + a}{r}$.

Article 47 (page 147)

1. 9; 36.
3. $2\frac{1}{2}$ hours.
5. 9 years.
7. 48 marbles.
9. $6\frac{2}{3}$ hours; $4\frac{1}{2}$ gallons.
11. $21\frac{9}{11}$ minutes past four o'clock.
13. 1:21 p.m.; the fighter plane flies 45 miles.
15. $3\frac{3}{4}$ gallons.
17. 5 quarts.
19. A, 19 days; B, 57 days; C, 76 days.

21. $8\frac{1}{3}$ days.
23. 4:50 p.m.
25. 60.
27. 21.
29. 12 nickels; 9 dimes, 8 quarters, 3 half dollars, 4 dollars. **31.** $27.
33. 2 parts 25-cent coffee; 1 part 40-cent coffee. **35.** 180 leaps.
37. 34.5 pounds of cream.
39. $\frac{s}{t+1}; \frac{st}{t+1}$.
41. $\frac{f-ns}{n-1}$ years.
43. $\frac{ab}{a+b}$ days.
45. $\frac{300(c-b)}{300-b}$ feet.
47. $\frac{n(b-c)}{b-a}$ pounds of tea selling at a cents per pound; $\frac{n(c-a)}{b-a}$ pounds of tea selling at b cents per pound.
49. $\frac{1200d}{a}$ oranges.

Article 48 (page 152)

1. 6 feet from 80-pound child.
3. 604 pounds.
5. 4 feet from 156-pound boy.
7. 3 feet from light end.
9. $5\frac{7}{13}$ feet from 60-pound weight.

CHAPTER VIII

Article 50 (page 162)

1. (2, 3).
3. $(\frac{2}{3}, \frac{3}{4})$.
5. (3, −7).
7. (−2, 5).
9. $(-2, \frac{6}{5})$.
11. $(\frac{20}{13}, \frac{35}{13})$.
13. Dependent equations.
15. $(-\frac{11}{31}, \frac{39}{31})$.
17. $(\frac{8}{5}, \frac{1}{10})$.
19. (−1, −5).
21. Dependent equations.
23. $(\frac{1}{2}, \frac{1}{3})$.
25. (2, 3).
27. $[\frac{1}{2}(a+b), \frac{1}{2}(a-b)]$.
29. (a, b).
31. $[(a+b), (a-b)]$.
33. $(\frac{8}{5}, -\frac{10}{3})$.
35. $(\frac{1}{3}, \frac{1}{5})$.
37. $(\frac{2}{3}, -\frac{3}{4})$.
39. (a) $a = l - (n-1)d; s = nl - \frac{nd}{2}(n-1)$.

(b) $a = \frac{s}{n} - \frac{(n-1)d}{2}; l = \frac{s}{n} + \frac{(n-1)d}{2}$.

(c) $n = \frac{l-a+d}{d}; s = \frac{(a+l)(l-a+d)}{2d}$.

(d) $a = \frac{2s}{n} - l; d = \frac{2(nl - s)}{n(n-1)}$.

(e) $l = \frac{2s}{n} - a; d = \frac{2(s - an)}{n(n-1)}$. (f) $s = \frac{n}{2}(a + l); d = \frac{l - a}{n - 1}$.

41. 14, 21. **43.** 37, 73. **45.** $\frac{5}{9}$.

47. Son's age is 5 years; father's age is 50 years. **49.** 12 m.p.h., rate of rowing in still water; 4 m.p.h., rate of current.

51. $\frac{7}{18}$ of a gallon from the first; $\frac{11}{18}$ of a gallon from the second.

53. 566 adults; 120 children. **55.** Dresses, \$20 each; coats, \$40 each.

57. 1 hour and $5\frac{5}{11}$ minutes. **59.** $16\frac{1}{2}$ years.

61. (a) $7x + 5y + 1 = 0$; (b) $x - 2y - 7 = 0$.

Article 51 (page 174)

1. $(-1, 2, 2)$. **3.** $(2, 4, 5)$. **5.** $(-2, -5, 3)$.

7. $(5, -2, 4)$. **9.** $(2, -3, -1)$. **11.** $(\frac{2}{3}, \frac{1}{2}, \frac{3}{4})$.

13. $(-3, 2, -1)$. **15.** $(1, 0, 0)$. **17.** $(\frac{1}{5}, -1, -\frac{1}{2})$.

19. $(-2, 3, -4)$. **21.** $(1, -2, 3)$. **23.** $(\frac{2}{3}, \frac{3}{4}, -\frac{2}{5})$.

25. $(6, 10, 15)$. **27.** A, \$4.50; B, \$3.75; C, \$5.25.

29. $(2, 10, 12)$. **31.** $(10, 20, 30)$.

33. 1 dog, 19 cats, 80 rabbits.

35. 200 miles level; 90 miles uphill; 60 miles downhill.

CHAPTER IX

Article 52 (page 177)

1. $\frac{4}{5}$. **3.** 2. **5.** 2. **7.** $\frac{1}{2}$.

9. $\frac{5}{4}$. **11.** π. **13.** 12. **15.** 15.

17. ab. **19.** b^2. **21.** 6. **23.** 8.

25. $\frac{1}{6}$. **27.** $\frac{1}{2}$.

29. 12 inches; 16 inches. **31.** 8 inches; 16 inches.

33. 1 inch; $\frac{5}{3}$ inches; $\frac{7}{3}$ inches. **35.** 6 inches; 8 inches.

37. $157\frac{1}{2}$ miles. **39.** 136 feet.

Article 53 (page 182)

3. $V = khr^2$. **5.** $F = kAV^2$. **7.** $R = k\frac{l}{A}$. **9.** $N = \frac{k}{\sqrt{l}}$.

11. 24. **13.** $18\frac{2}{3}$. **15.** $z = \frac{4x}{y}$. **17.** $V = 4.1888r^3$.

19. B.H.P. $= 0.4d^2n$.

21. $6\frac{3}{16}$ centimeters.

23. 9 days.

25. 31,250 feet; $7812\frac{1}{2}$ feet.

27. $266\frac{2}{3}$ pounds; $4571\frac{3}{7}$ pounds.

29. $27\frac{9}{13}$ pounds; 40 pounds.

31. 1029.12 feet; 2315.52 feet.

33. 0.272 inch.

35. $16\sqrt{5}$ feet.
37. $5\sqrt{2}$ feet.
39. 990 miles.
41. 5.2 times as far.
43. $333\frac{1}{3}$ revolutions per minute.
45. 140 pounds.

CHAPTER X

Article 57 (page 193)

1. (a) $3 - 2i$; (b) $5 + 2i$; (c) $-7 - i\sqrt{3}$; (d) $0 + 2i\sqrt{6}$; (e) $8 + 0i$; (f) $\sqrt{7} + 0i$; (g) $1 + 0i$; (h) $0 + 0i$; (i) $0 + i$.
3. $-6 + 11i$.
5. -6.
7. $4 - 7i$.
9. $-i$.
11. $-i$.
13. $-8i$.
15. i.
17. $-\frac{3}{2}i$.
19. $5 - 5i$.
21. $1 + 3i$.
23. $-36 - 9i$.
25. $7 + 17i$.
27. 0.
29. $\frac{4}{13} - \frac{7}{13}i$.
31. $-\frac{18}{25} + \frac{i}{25}$.
33. $\frac{63}{25} + \frac{16}{25}i$.
35. $\frac{1 - 2\sqrt{6}}{8} + \frac{\sqrt{3} + 2\sqrt{2}}{8}i$.
37. ∞.
39. 0.
41. 0.
43. Yes.
45. Yes.

Article 59 (page 203)

1. $4\sqrt{2} + 4i\sqrt{2}$.
3. $6 - 6i\sqrt{3}$.
5. $2 + 2i\sqrt{3}$.
7. $\frac{3}{2} - \frac{3}{2}i\sqrt{2}$.
9. $-4\sqrt{3} - 4i$.
11. $\frac{1}{128} - \frac{\sqrt{3}}{128}i$.
13. $-2 + 2i$.
15. $\frac{\sqrt{3}}{2} + \frac{i}{2}$.
17. $-2\sqrt{3} - 2i$.
19. $\frac{1}{2} + \frac{1}{2}i$.
21. $\frac{1}{8}$.
23. $4\sqrt{3} + 4i$.
25. $\frac{\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}; -\frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}$.
27. $2; -1 + i\sqrt{3}; -1 - i\sqrt{3}$.
29. $\sqrt{3} + i; -1 + i\sqrt{3}; -\sqrt{3} - i; 1 - i\sqrt{3}$.
31. $2(\cos 18° + i \sin 18°)$; $2i$; $2(\cos 162° + i \sin 162°)$; $2(\cos 234° + i \sin 234°)$; $2(\cos 306° + i \sin 306°)$.
33. $1; -\frac{1}{2} + \frac{\sqrt{3}}{2}i; -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.
35. $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i; -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i; -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i; \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.
37. $3; -\frac{3}{2} + \frac{3\sqrt{3}}{2}i; -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$.

39. $\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i; -\frac{3\sqrt{2}}{2} + \frac{3\sqrt{2}}{2}i; -\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i;$

$\frac{3\sqrt{2}}{2} - \frac{3\sqrt{2}}{2}i.$

41. $\frac{3}{2} + \frac{3\sqrt{3}}{2}i; -3; \frac{3}{2} - \frac{3\sqrt{3}}{2}i.$

43. $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i; \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i.$

CHAPTER XI

Article 60 (page 208)

1. $2; -1.$
3. $5; 5.$
5. $-\frac{3}{4}; -\frac{3}{2}.$
7. $-3; -5.$
9. $4; -4.$
11. $0; \frac{7}{6}.$
13. $1; -\frac{5}{3}.$
15. $0; \frac{5}{9}.$
17. $-1; \frac{7}{2}.$

Article 60 (page 213)

1. $-3 \pm \sqrt{2}.$
3. $\frac{-5 \pm \sqrt{13}}{6}.$
5. $3 \pm i.$
7. $\frac{3 \pm \sqrt{5}}{2}.$
9. $\frac{1 \pm \sqrt{41}}{4}.$
11. $\frac{2 \pm i\sqrt{2}}{3}.$
13. $9; -7.$
15. $a; \frac{1}{a}.$
17. $\pm\sqrt{-\frac{c}{a}}.$
19. $1; -a.$
21. $a; -2.$
23. $4y; -\frac{5}{2}y.$
25. $-y; \frac{5y}{3}.$
27. $\frac{5 \pm i\sqrt{71}}{6}y.$
29. $2; -\frac{7}{2}.$
31. $-b + a; -b - a.$
33. $1; -1 - \sqrt{3}.$
35. $\frac{9 \pm i\sqrt{303}}{24}.$
37. $7; -2.$
39. $a; \frac{1}{a}.$
41. $\frac{3 \pm \sqrt{5}}{2}.$
43. $3; \frac{3}{5}.$
45. $\frac{11}{3}.$
47. $-1; \frac{14}{5}.$
49. $h = \frac{V}{\pi r^2}; r = \pm\sqrt{\frac{V}{\pi h}}.$
51. $X = \pm\sqrt{\frac{R}{g} - R^2}; R = \frac{1 \pm \sqrt{1 - 4X^2R^2}}{2g}.$
53. $a = \frac{S}{n} - \frac{(n-1)d}{2}; d = \frac{2(S - na)}{n(n-1)};$

$n = \frac{(d - 2a) \pm \sqrt{(2a - d)^2 + 8dS}}{2d}.$

55. $W = \frac{4m}{3DL^2}$; $D = \frac{4m}{3WL^2}$; $L = \pm \frac{2\sqrt{3mWD}}{3WD}$.

57. $v = \frac{s}{t} - \frac{1}{2}gt$; $g = \frac{2(s - vt)}{t^2}$; $t = \frac{-v \pm \sqrt{v^2 + 2gs}}{g}$.

59. $m = \frac{\pi k r^2 s}{gl}$; $l = \frac{\pi k r^2 s}{mg}$; $r = \pm \sqrt{\frac{mgl}{\pi k s}}$.

61. 10.

63. 10; 11.

65. 7; 8.

67. 10 feet by 15 feet.

69. 40 miles per hour; 25 miles per hour.

71. $200.

73. 360 eggs.

75. 5 feet wide.

77. 11 towns.

79. $3\frac{1}{2}$ inches.

81. $V_0 - gt$; $V_0 + gt$.

83. $\frac{2}{3}$ of a mile.

85. 5 seconds; $\frac{2}{9}$ of a mile.

87. No solution.

89. $\frac{-(a + b) + \sqrt{a^2 + 6ab + b^2}}{4}$ feet.

91. $\frac{d + \sqrt{d^2 + r^2t^2}}{t}$ miles per hour.

93. $\frac{n}{\sqrt{2} - 1}$ feet.

95. $\frac{d(d - 2h)}{h}$.

97. $\frac{2d - b + \sqrt{b^2 + 4d^2}}{2}$ days.

Article 61 (page 220)

1. 484.

3. 4; 16.

5. $\frac{1}{16}$.

7. 3; 2.

9. 3.

11. 4.

13. 2.

15. 6.

17. 4.

19. $\frac{1}{8}$; -27.

21. 9; -3.

23. 1; $\frac{1}{4}$.

25. ± 4; ± 5.

27. 1; 2; -2; -3.

29. $-\frac{1}{2}$; -4.

Article 63 (page 226)

1. $\frac{13}{20}$; 12.

3. 65; -350.

5. $-\frac{8}{3}$; $-\frac{50}{3}$.

7. $\frac{2}{3}$; $-\frac{1}{3}$.

9. -1; $-\frac{7}{k}$.

11. $3 + k$; $4 - 2k$.

13. Real.

15. Real.

17. Real.

19. Real.

21. Imaginary.

23. Imaginary.

25. $k = \pm 6$; $k \leqq -6$; $k \geqq 6$; $-6 < k < 6$.

27. $k = \frac{1}{5}$; $k \leqq \frac{1}{5}$; $k > \frac{1}{5}$.

29. $k = 6, -2$; $k \geqq 6, k \leqq -2$; $-2 < k < 6$.

31. $k = 16$; $k \leqq 16$; $k > 16$.

33. $k = \frac{17}{16}$; $k \leqq \frac{17}{16}$; $k > \frac{17}{16}$.

35. No values; all values; no values.

37. $k = 4$; all values; no values.

39. $k = \pm \frac{7}{6}$; $-\frac{7}{6} \leqq k \leqq \frac{7}{6}$; $k < -\frac{7}{6}$; $k > \frac{7}{6}$.

41. 5. **43.** 1. **45.** -6. **47.** 1.
49. $\frac{25}{4}$. **51.** 3. **53.** $k < -\frac{1}{4}$. **55.** $-\frac{1}{5}$.
57. $x^2 + 2x - 3 = 0$. **59.** $x^2 + 3x - 10 = 0$.
61. $abx^2 - (a^2 + b^2)x + ab = 0$. **63.** $ax^2 - (a^2 + 1)x + a = 0$.
65. $20x^2 + x - 12 = 0$. **67.** $2x^2 - 2x - 1 = 0$.
69. $x^2 - 6x + 10 = 0$.
71. (a) $ax^2 - bx + c = 0$; (b) $ax^2 + 2bx + 4c = 0$;
(c) $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$; (d) $cx^2 + bx + a = 0$.
73. $x > 2$. **75.** $-3 < x < 3$. **77.** $-3 < x < \frac{2}{3}$.
79. $x > 3$ and $x < -\frac{2}{3}$. **81.** $x > 3$ and $x < -3$.
83. $x > 5$. **85.** $x > -8$. **87.** $x < \frac{5}{4}$.

CHAPTER XII

Article 68 (page 237)

1. (a) $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10; \pm 20$; (b) $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24$; (c) $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$; (d) $\pm 1, \pm 2, \pm \frac{1}{2}$; (e) $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm \frac{1}{2}, \pm \frac{3}{2}$; (f) $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{1}{6}$.
3. 0. **5.** $2a^n$. **7.** Yes. **9.** No. **11.** $-\frac{5}{2}$.

Article 75 (page 252)

1. (a) 2 positive, 2 negative, 2 imaginary; (b) no positive, 2 negative, 6 imaginary; (c) 1 positive, 2 negative, 4 imaginary; (d) 1 positive, no negative, 2 imaginary.
3. 2, 5, -2. **5.** 3, 4, -2. **7.** 1, -3, -4.
9. 3, -1, -2. **11.** 0, 2, -3, -6. **13.** 2, 2, -3, -3.
15. $2, \frac{-1 \pm i\sqrt{3}}{2}$. **17.** $-3, 1 \pm i\sqrt{3}$. **19.** $-3, \pm\sqrt{2}$.
21. $-2, \frac{5 \pm \sqrt{21}}{2}$. **23.** $3, \frac{-1 \pm \sqrt{5}}{2}$. **25.** $-2, -4, -6$.
27. $\frac{1}{2}, 3, -\frac{2}{3}$. **29.** $2, -\frac{1}{2}, -3$. **31.** $\frac{4}{3}, \frac{3}{2}, -2$.
33. $\frac{5}{2}, -1, \frac{1 \pm i\sqrt{11}}{3}$. **35.** $-\frac{5}{2}, \frac{5 \pm \sqrt{57}}{4}$. **37.** $\frac{2}{3}, -2, \frac{2 \pm i\sqrt{2}}{3}$.
39. $\frac{1}{2}, \frac{3}{2}, \pm 2\sqrt{2}$. **41.** $1, 1, -\frac{2}{3}$. **43.** $4, -5, -6$.
45. $2, 6, -3, -5$. **47.** $-6, -10, 8 \pm \sqrt{58}$.

Article 76 (page 258)

1. $x^3 - 17x^2 - 30x + 216 = 0$. **3.** $x^3 + 3x^2 - 22x - 24 = 0$.
5. $3x^4 - 40x + 16 = 0$. **7.** $54x^3 - 9x^2 + 9x - 5 = 0$.
9. $6x^5 - 4x^3 + 6x^2 + x + 5 = 0$. **11.** $4x^4 + 2x^3 - 3x^2 + 2 = 0$.

13. $x^3 + 4x^2 - 3 = 0$. **15.** $x^3 + 3x^2 - 6x + 7 = 0$.
17. $2x^3 + 10.2x^2 + 15.84x + 7.776 = 0$.
19. $x^3 - 5x^2 + 10x - 10 = 0$.
21. 2, 6, −3, −5. **23.** 3, −4, $\pm 5i$.
25. $-6, -10, 8 \pm \sqrt{58}$. **27.** $-\frac{7}{3}, \pm 2\sqrt{3}$. **29.** $\frac{8}{3}, \frac{10}{3}, -5$.

Article 77 (page 271)

1. 2, 3.791, −0.791.
3. 1.817.
5. 1.423, 5.247, −0.669.
7. 3.137, −2.5, −0.637.
9. 0.441, 2.398, −2.838.
11. 2.180, −4.691, −0.489.
13. −1.649.
15. 2, $-\frac{3}{2}$, 1.414, −1.414.
17. $-\frac{2}{3}$, 1.182, 2.873.
19. 2.201, 2.779, −1.227.
21. 0.680, 4.917, −0.598.
23. 1.294, 4.100, −3.394.
25. 1.202, 3.128, −1.330.
27. −2.524.
29. 1, 1.355.
31. 1.125.
33. (a) 16.52 inches; (b) 1 foot; (c) 8.554 inches; (d) 2.805 inches in radius. **35.** 6.380 feet.

Article 79 (page 277)

1. $3.522, -1.761 \pm 0.553i$.
3. $4.522, 0.761 \pm 0.553i$.
5. $-1.651, -0.175 \pm 1.547i$.
7. $-1.339, 0.969 \pm 0.506i$.
9. 2, 2, −4.
11. 1, 1, −5.
13. 3, −1, −2.
15. 3.137, −2.5, −0.637.
17. 0.680, 4.917, −0.598.
19. 2.180, −0.489, −4.691.

Article 80 (page 280)

1. $\pm i\sqrt{2}, -2, \pm i$.
3. $4, 4, \pm i$.
5. 1, 1, −2, −4.
7. $2, 4, -1 \pm \sqrt{2}$.
9. $\frac{1 \pm i\sqrt{3}}{2}, \frac{3 \pm i\sqrt{7}}{2}$.
11. $\frac{1}{2}, -\frac{5}{2}, \pm\frac{i}{2}\sqrt{3}$.
13. $\frac{-3 \pm \sqrt{17}}{2}, \frac{-1 \pm i\sqrt{3}}{2}$.
15. $1, 1, 1 \pm \sqrt{2}$.
17. $1, 1, 2 \pm i$.
19. $\pm 2i, 1 \pm i\sqrt{2}$.

CHAPTER XIII

Article 82 (page 289)

1. (3, 4); (−4, −3).
3. (3, 2); (−5, 6).
5. (5, 2); (−2, −5).
7. $(3 + i, 3 - i); (3 - i, 3 + i)$.
9. $(0, -4); (\frac{11}{2}, \frac{3}{2})$.
11. $(2, 4); (\frac{1}{2}, -2)$.
13. $(\frac{3}{8}, -\frac{7}{4}); (-2, 3)$.
15. $(-5, 3); (-2, \frac{3}{2})$.
17. (−5, −3); (−5, −3).
19. (−2, 1); (4, −1).

Article 82 (page 292)

1. $(4, 3); (4, -3); (-4, 3); (-4, -3)$.

3. $(\sqrt{5}, \sqrt{5}); (\sqrt{5}, -\sqrt{5}); (-\sqrt{5}, \sqrt{5}); (-\sqrt{5}, -\sqrt{5})$.

5. $(9, 4i\sqrt{2}); (9, -4i\sqrt{2}); (-9, 4i\sqrt{2}); (-9, -4i\sqrt{2})$.

7. $(0, 2); (0, 2); (0, -2); (0, -2)$.

9. $(\sqrt{2}, \sqrt{5}); (\sqrt{2}, -\sqrt{5}); (-\sqrt{2}, \sqrt{5}); (-\sqrt{2}, -\sqrt{5})$.

11. $(3, 2); (3, -2); (-3, 2); (-3, -2)$.

Article 82 (page 297)

1. $(2, 3); (-2, -3); (\sqrt{6}, \sqrt{6}); (-\sqrt{6}, -\sqrt{6})$.

3. $(\frac{3}{2}\sqrt{10}, \frac{3}{10}\sqrt{10}); (-\frac{3}{2}\sqrt{10}, -\frac{3}{10}\sqrt{10}); (\frac{1}{2}\sqrt{10}, \frac{9}{10}\sqrt{10})$; $(-\frac{1}{2}\sqrt{10}, -\frac{9}{10}\sqrt{10})$.

5. $(2, 1); (-2, -1); (10, 7); (-10, -7)$.

7. $(2, 6); (-2, -6); (3, -5); (-3, 5)$.

9. $(3, -2); (-3, 2); (\frac{1}{3}i\sqrt{3}, \frac{4}{3}i\sqrt{3}); (-\frac{1}{3}i\sqrt{3}, -\frac{4}{3}i\sqrt{3})$.

11. $(2, 2); (-2, -2); (3, \frac{3}{2}); (-3, -\frac{3}{2})$.

13. $(5, -3); (-5, 3); (4, -\frac{3}{2}); (-4, \frac{3}{2})$.

15. $(5, 0); (-5, 0); (\frac{1}{2}\sqrt{10}, \frac{3}{5}\sqrt{10}); (-\frac{1}{2}\sqrt{10}, -\frac{3}{5}\sqrt{10})$.

17. $(2, 1); (-2, -1); (-3, 2); (3, -2)$.

Article 82 (page 298)

1. $(0, 3); (0, -3); (3, \frac{3}{2}); (-3, -\frac{3}{2})$. **3.** $(-1, -4); (2, -1)$.

5. $(0, 0); (0, 0); (-1, -2); (-2, -1)$. **7.** $(1, 1); (\frac{1}{9}, \frac{7}{9})$.

9. $(6, 3); (-4, -2); (-6, 2); (\frac{9}{2}, -\frac{3}{2})$.

11. $(2, 1); (-2, -1); (-3, 1); (\frac{1}{2}, -\frac{1}{6})$.

13. $(2, 4); (-5, -3)$. **15.** $(2, 5); (-4, 3)$.

17. $(1, 3); (3, -1); (-3, 1); (-3, -3)$. **19.** $(2, -5); (-3, 4)$.

21. $(-3, 4); (2, -3)$. **23.** 16 and 11; -16 and -11.

25. 18 rods by 62 rods. **27.** 8 feet by 10 feet.

29. A's rate is 40 miles per hour; B's rate is 32 miles per hour. **31.** 60 dozen; 27 cents per dozen.

33. 15 members. **35.** 32 rods by 40 rods.

37. 36 miles per hour; 20 miles. **39.** $7\frac{1}{5}$.

41. A paid 40 cents per pound; B paid 25 cents per pound.

CHAPTER XIV

Article 83 (page 305)

1. $l = 73; s = 570$.

3. $l = -47; s = -392$.

5. $l = 4\frac{1}{2}; s = 58\frac{1}{3}$.

7. $n = 5; d = 10$.

9. $a = -80; s = -350$.

11. $l = \dfrac{2s}{n} - a$.

13. $l = \frac{s}{n} + \frac{(n-1)d}{2}$. **15.** 30.

17. 210; 5,050.

19. $27\frac{8}{11}$, $25\frac{5}{11}$, $23\frac{2}{11}$, $20\frac{10}{11}$, $18\frac{7}{11}$, $16\frac{4}{11}$, $14\frac{1}{11}$, $11\frac{9}{11}$, $9\frac{6}{11}$, $7\frac{3}{11}$.

21. $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$, 1, $\frac{7}{6}$, $\frac{4}{3}$. **23.** n^2.

25. 49,500. **27.** 1,610 feet. **29.** 200; \$6,010.

Article 85 (page 313)

1. $l = 512$; $s = 1{,}022$.

3. $l = \frac{1}{128}$; $s = \frac{1023}{128}$.

5. $l = 4 \cdot 10^{-9}$; $s = 4.444444444$.

7. $a = \frac{2}{27}$; $s = 2186\frac{26}{27}$.

9. $r = 0.1$; $s = 111.111$.

11. $r = \frac{s-a}{s-l}$.

13. $l = \frac{s(1-r)r^{n-1}}{1-r^n}$.

15. ar^{11}, ar^{189}, ar^{k-1}, ar^k, ar^{n-6}.

17. 16.

19. 30, 150.

21. $2^{1/8}$, $2^{1/4}$, $2^{3/8}$, $2^{1/2}$, $2^{5/8}$, $2^{3/4}$, $2^{7/8}$.

23. $a^{5/6}b^{1/6}$, $a^{3/6}b^{3/6}$, $a^{1/2}b^{1/2}$, $a^{3/6}b^{3/6}$, $a^{1/6}b^{5/6}$.

25. \$222,222,222.20.

27. 1.84×10^{19}.

29. 4,096.

31. 24.

33. $\frac{3}{2}$.

35. $\frac{10}{9}$.

37. $\frac{1}{1-x}$.

39. $\frac{2}{9}$.

41. $\frac{5}{9}$.

43. .02.

45. $1\frac{34}{99}$.

47. $\frac{77}{90}$.

49. $414\frac{7}{12}$ feet.

Article 86 (page 315)

1. $\frac{12}{5}$, 2, $\frac{12}{7}$, $\frac{3}{2}$.

3. $\frac{2}{3}$, $\frac{4}{7}$, $\frac{1}{2}$, $\frac{4}{9}$.

5. $-\frac{1}{5}$, $-\frac{1}{8}$, $-\frac{1}{11}$, $-\frac{1}{14}$.

7. $\frac{6}{17}$, $\frac{3}{11}$, $\frac{2}{9}$, $\frac{3}{16}$.

9. $\frac{8}{7}$, $\frac{4}{3}$, $\frac{8}{5}$.

11. $\frac{50}{33}$, $\frac{25}{28}$, $\frac{50}{79}$, $\frac{25}{51}$.

13. $\frac{9}{62}$, $\frac{9}{52}$, $\frac{3}{14}$, $\frac{9}{22}$.

CHAPTER XV

Article 88 (page 322)

1. 120. **3.** 840. **5.** 243.

7. 48; 72. **9.** 35. **11.** 36.

13. 240. **15.** 3^{12}. **17.** 25.

19. 3,360; 34,650; 64,864,800. **21.** 720.

23. No. **25.** 36. **27.** $n(n-1)(n-2)(n-3)(n-4)$.

29. $n(n-1)(n-2)\cdots(n-r+2)$.

31. $(n-3)(n-4)(n-5)$. **33.** $\frac{40!}{3!}$. **35.** 56.

37. 8. **39.** $\frac{9}{7}$. **41.** $(n-1)[(n-1)!]$.

Article 89 (page 328)

1. 924. **3.** 22,100. **5.** 560.
7. 20,475. **9.** 210. **11.** 45.
13. 3,500. **15.** 49,728. **17.** 560(8!).
19. 31. **21.** 21.
23. (a) 140; (b) 56; (c) 196.
25. $\dfrac{52!}{(13!)^4}$. **27.** 40,320. **29.** 8,064,000.
31. $\dfrac{n(n-1)(n-2)}{3!}$. **33.** $\dfrac{(n-2)(n-3)(n-4)(n-5)(n-6)}{5!}$.
35. $\dfrac{(n-1)(n-2)(n-3)\cdots(n-r+1)}{(r-1)!}$.

Article 91 (page 336)

1. (a) $\frac{3}{8}$; (b) $\frac{5}{8}$.
3. $\frac{20}{221}$.
5. (a) $\frac{2}{25}$; (b) $\frac{1}{300}$.
7. $\frac{1}{120}$.
9. (a) $\frac{1}{8}$; (b) $\frac{1}{8}$.
11. (a) $\frac{5}{16}$; (b) $\frac{1}{2}$.
13. $\frac{3}{100}$.
15. $\frac{91}{216}$.
17. $\frac{1}{6}$.
19. $\frac{56}{495}$.
21. ${}_{10}C_5\left(\dfrac{15}{64}\right)^5\left(\dfrac{49}{64}\right)^5$.
23. $\frac{325}{833}$.
25. (a) $\frac{1}{364}$; (b) $\frac{1}{26}$; (c) $\frac{1}{364}$; (d) $\frac{2}{13}$; (e) $\frac{4}{91}$; (f) $\frac{3}{13}$.
27. $\frac{1}{720}$.
29. $\frac{14}{15}$, $\frac{1}{5}$.
31. $\frac{93}{260}$.
33. $\dfrac{16(39!)(13!)}{52!}$ or 2.52×10^{-11}.
35. 0.93; 0.85; 0.76; 0.63; 0.54.
37. 0.664; 0.302.
39. 0.133; 0.211.

CHAPTER XVI

Article 94 (page 346)

1. $a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$.
3. $x^7 + 14x^6 + 84x^5 + 280x^4 + 560x^3 + 672x^4 + 448x + 128$.
5. $x^{12} - 6x^{10}y^2 + 15x^9y^4 - 20x^6y^6 + 15x^4y^8 - 6x^2y^{10} + y^{12}$.
7. $a^{10} - 5a^8b + 10a^6b^2 - 10a^4b^3 + 5a^2b^4 - b^5$. **9.** 1,419,857.
11. $\dfrac{a^8}{b^8} + 8\dfrac{a^9}{b^9} + 28\dfrac{a^{10}}{b^{10}} + 56\dfrac{a^{11}}{b^{11}} + 70\dfrac{a^{12}}{b^{12}} + 56\dfrac{a^{13}}{b^{13}} + 28\dfrac{a^{14}}{b^{14}} + 8\dfrac{a^{15}}{b^{15}} + \dfrac{a^{16}}{b^{16}}$.
13. $\frac{1}{128}x^{14}y^{-7} + \frac{7}{32}x^{10}y^{-4} + \frac{21}{8}x^6y^{-1} + \frac{35}{2}x^2y^2 + 70x^{-2}y^5 + 168x^{-6}y^8 + 224x^{-10}y^{11} + 128x^{-14}y^{14}$.
15. $x^3 - 6x^{5/2}y^{1/2} + 15x^2y - 20x^{3/2}y^{3/2} + 15xy^2 - 6x^{1/2}y^{5/2} + y^3$.
17. $64x^6y^{-12} - 96x^4y^{-8} + 60x^2y^{-4} - 20 + \frac{15}{4}x^{-2}y^4 - \frac{3}{8}x^{-4}y^8 + \frac{1}{64}x^{-6}y^{12}$.

19. $x^3 - 6x^{13/6} + 15x^{4/3} - 20x^{1/2} + 15x^{-1/3} - 6x^{-7/6} + x^{-2}$.
21. $\frac{32}{243} - \frac{80}{81}x^{-1} + \frac{80}{27}x^{-2} - \frac{40}{9}x^{-3} + \frac{10}{3}x^{-4} - x^{-5}$.
23. $x^{-6} + 6x^{-5}y^{-2} + 15x^{-4}y^{-4} + 20x^{-3}y^{-6} + 15x^{-2}y^{-8} + 6x^{-1}y^{-10} + y^{-12}$.
25. $x^{20} + 20x^{19}a + 190x^{18}a^2 + 1140x^{17}a^3 + 4845x^{16}a^4 + \cdots$.
27. $x^{12}y^{-12} - 12x^{10}y^{-10} + 66x^8y^{-8} - 220x^6y^{-6} + 495x^4y^{-4} - \cdots$.
29. $x^{2n} + 2nx^{2n-1}a + n(2n-1)x^{2n-2}a^2 + \frac{1}{3}n(2n-1)(2n-2)x^{2n-3}a^3 + \frac{1}{6}n(2n-1)(n-1)(2n-3)x^{2n-4}a^4 + \cdots$.
31. $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots$.
33. $x - \frac{1}{4}x^{-1} - \frac{1}{32}x^{-3} - \frac{1}{128}x^{-5} - \frac{5}{2048}x^{-7} - \cdots$.
35. $x^{-2} - 2x^{-3}a^{1/2} + 3x^{-4}a - 4x^{-5}a^{3/2} + 5x^{-6}a^2 - \cdots$.
37. $1 - x + x^2 - x^3 + x^4 - \cdots$.
39. $x^{2/3} - \frac{2}{3}x^{-1/3}a - \frac{1}{9}x^{-4/3}a^2 - \frac{4}{81}x^{-7/3}a^3 - \frac{7}{243}x^{-10/3}a^4 - \cdots$.
41. $x^{-1} + \frac{1}{2}a^2x^{-3} + \frac{3}{8}a^4x^{-5} + \frac{5}{16}a^6x^{-7} + \frac{35}{128}a^8x^{-9} + \cdots$.
43. $7{,}920x^8y^4$. **45.** $15{,}890{,}700a^6x^{44}$. **47.** $560a^{3/2}b^{13/2}$.
49. $\frac{1215}{256}x^7$. **51.** $-\frac{21}{1024}x^6$. **53.** $x^{-24}a^{22}$.
55. $-\frac{32}{81}a^9x^{-5}$. **57.** $-\frac{63}{8}x^{10}a^{10}$.
59. $\dfrac{n(n-1)\cdots(n-r+1)}{r!}a^{n-r}x^r$.
61. $\dfrac{n(n-1)\cdots(n-r+4)}{(r-3)!}a^{n-r+3}x^{r-3}$.
63. 194,481. **65.** 92,236,816.
67. 10.10. **69.** 4.98. **71.** 2.03.

CHAPTER XVII

Article 96 (page 359)

1. 1. **3.** 11.
5. $(a-b)(b-c)(c-a)$. **7.** $4abc$.
9. 1. **11.** -91.
13. 5. **15.** 114.
17. $(b-a)^3$. **19.** 1.
21. (a) $(2, -3), (-4, 5)$; (b) $(-\frac{1}{2}, \frac{3}{2}), (\frac{2}{3}, -\frac{4}{3})$.
23. (a) $\frac{3}{2}$; (b) 1; (c) 1, 3.
25. $w = -1; x = 1; y = 2; z = -3$.
27. $w = 1; x = 0; y = 0; z = -1$.
29. $w = 1; x = 2; y = 2; z = 1$.
31. (a) $x + 5y - 17 = 0$; (b) $4x + 5y + 12 = 0$;
(c) $x + 2y + 7 = 0$. **33.** (a) $x^2 + y^2 - x + y - 2 = 0$;
(b) $13x^2 + 23y^2 - 25x - 19y - 42 = 0$; (c) $y^2 - 2x + y + 4 = 0$;
(d) $x^2 + 6x - 4y - 7 = 0$.

CHAPTER XVIII

Article 97 (page 367)

1. $\frac{2}{x-1}+\frac{3}{x-2}$.

3. $-\frac{1}{x+2}+\frac{2x+1}{x^2+1}$.

5. $\frac{4}{2x+5}-\frac{2}{3x-2}$.

7. $\frac{2}{x+3}+\frac{3}{(x-2)^2}$.

9. $\frac{3}{x+3}+\frac{2}{x-2}+\frac{4}{(x-2)^2}$.

11. $\frac{3}{2-x}+\frac{2x-3}{3x^2+2}$.

13. $\frac{3x+2}{2x^2+3}-\frac{4}{(x-2)^2}$.

15. $\frac{x+2}{x^2-2x+3}+\frac{x-5}{(x^2-2x+3)^2}$.

17. $-\frac{2}{3x+1}+\frac{4}{2x-1}$.

19. $-\frac{1}{x+2}+\frac{2x-4}{x^2+1}-\frac{2x-4}{(x^2+1)^2}$.

21. $\frac{3}{2x-3}-\frac{3}{3x+2}$.

23. $x^2+1-\frac{2}{3x+2}+\frac{3}{2x-3}$.

25. $\frac{2}{x-2}+\frac{3}{(x-2)^2}$.

27. $\frac{3}{(x-2)^2}-\frac{2}{x-2}$.

29. $\frac{2}{x-2}+\frac{3}{(x-2)^2}-\frac{4}{x-1}$.

31. $\frac{3x-2}{x^2+x+1}-\frac{2}{2x+3}$.

33. $-\frac{2}{x-1}+\frac{2x+3}{x^2+1}+\frac{5x-3}{(x^2+1)^2}$.

35. $\frac{3x}{2x^2+1}-\frac{2}{x-2}+\frac{3}{(x-2)^2}$.

37. $\frac{2x+3}{2x^2-x+1}-\frac{2}{2x-1}-\frac{2}{(2x-1)^2}$.

39. $\frac{1}{2a(x-a)}-\frac{1}{2a(x+a)}$.

41. $-\frac{3}{x}+\frac{2}{x^2}+\frac{2}{x^3}+\frac{7x-7}{2x^2-x+1}$.

43. $\frac{3}{x-2}-\frac{3x+2}{x^2+1}+\frac{2x-1}{(x^2+1)^2}$.

CHAPTER XIX

Article 100 (page 374)

1. (a) $\frac{3}{4}$; (b) $\frac{1}{20}$; (c) $\frac{2}{5}$; (d) $\frac{9}{8}$.

3. (a) 50%; (b) 125% (c) 200%.

5. \$16.88. **7.** \$1.31. **9.** 5%. **11.** \$1,231.53.

13. (a) \$6,036.60; (b) \$17,246.44. **15.** (a) \$258.43; (b) \$570.27.

17. (a) 6.09%; (b) 6.136%; (c) 6.168%.

19. (a) 5.063%, (b) 5.095%; (c) 5.116%.

21. (a) 5.913%; 3.961%; 1.990%; (b) 5.870%; 3.941%; 1.985%; (c) 5.841%; 3.928%; 1.982%. **23.** 175 per 1,000. **25.** Electricity 64.27%; gas 29.67%; steam heat 6.06%. **27.** 1.34%. **29.** 23.78% loss; 1.22% gain; 56.86% gain; 10.26% loss; 26.67% loss; 900% gain. **31.** \$2,149.02. **33.** 11.88 years; 14.21 years; 11.72 years; 14.03 years. **35.** \$623.17.

Article 101 (page 381)

The following answers were obtained by using a seven-place table of logarithms.

1. \$18,392.80.
3. \$795.05.
5. \$5,386.19.
7. \$151.55.
9. \$99.95.
11. \$4,555.53.
13. \$18,000.00.
15. \$8,023.58.
17. \$274.12.
19. \$136.10.
21. \$33.58.
23. \$1,840.46.
25. \$112.12.
27. \$5,518.71.

CHAPTER XX

Article 102 (Page 387)

1. $\frac{1}{n(n+1)}$; $\frac{1}{k(k+1)}$; $\frac{1}{(k+1)(k+2)}$.

3. $\frac{1}{2n-1}$; $\frac{1}{2k-1}$; $\frac{1}{2k+1}$.

5. $\frac{1}{\sqrt[n+1]{2}}$; $\frac{1}{\sqrt[k+1]{2}}$; $\frac{1}{\sqrt[k+2]{2}}$.

7. $\frac{n(n+1)}{(2n+1)(2n+3)}$; $\frac{k(k+1)}{(2k+1)(2k+3)}$; $\frac{(k+1)(k+2)}{(2k+3)(2k+5)}$.

9. $\frac{x^n}{2n}$; $\frac{x^k}{2k}$; $\frac{x^{k+1}}{2k+2}$.

11. $(-1)^{n-1}n^2x^{n-1}$; $(-1)^{k-1}k^2x^{k-1}$; $(-1)^k(k+1)^2x^k$.

13. $(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$; $(-1)^{k-1}\frac{x^{2k-1}}{(2k-1)!}$; $(-1)^k\frac{x^{2k+1}}{(2k+1)!}$.

15. $2+\frac{3}{2!}+\frac{4}{3!}+\frac{5}{4!}$.

17. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}$.

19. $\frac{1\cdot 2^2}{1!}+\frac{2\cdot 3^2}{3!}+\frac{3\cdot 4^2}{5!}+\frac{4\cdot 5^2}{7!}$.

21. $-\frac{3\cdot 1^2}{2\cdot 1}+\frac{3\cdot 2^2}{2^2\cdot 3}-\frac{3\cdot 3^2}{2^3\cdot 5}+\frac{3\cdot 4^2}{2^4\cdot 7}$.

23. $\frac{1\cdot 2}{1\cdot 1}+\frac{2\cdot 3}{3\cdot 3^2}+\frac{3\cdot 4}{3^2\cdot 5^2}+\frac{4\cdot 5}{3^3\cdot 7^2}$.

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CHAPTER XXI

Article 107 (page 403)

1. Convergent.
3. Convergent.
5. Convergent.
7. Absolutely convergent.
9. Divergent.
11. Divergent.
13. Divergent.
15. Convergent.
17. Convergent.
19. Absolutely convergent.
21. For all real values of x; written symbolically $-\infty < x < \infty$.
23. $-1 < x < 1$.
25. $-1 \leqq x < 1$.
27. $-\infty < x < \infty$.
29. $-3 < x < 3$.
31. $-1 \leqq x \leqq 1$.
33. $-\frac{2}{3} < x < \frac{2}{3}$.
35. $-\infty < x < \infty$.
37. $-\infty < x < \infty$.
39. $-\infty < x < \infty$.
41. $-1 \leqq x \leqq 1$.
43. Convergent.

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